MODELING DEPENDENCY RELATIONSHIPS WITH COPULAS

Joseph Atwood
jatwood@montana.edu
and
David Buschena
buschena.@montana.edu

SCC-76 Annual Meeting, Gulf Shores, March 2007

MONTANA STATE UNIVERSITY

REINSURANCE COMPANY REQUIREMENT

- Considering reinsuring a particular product
- No disagreement with producer level rating procedures
  - Yield distributions
  - Quality distributions
- Company required estimates of VAR (Value at Risk)
  - 1% and 5%
  - Account for dependencies in yield and quality across producers
  - Yields and quality realizations not normally distributed

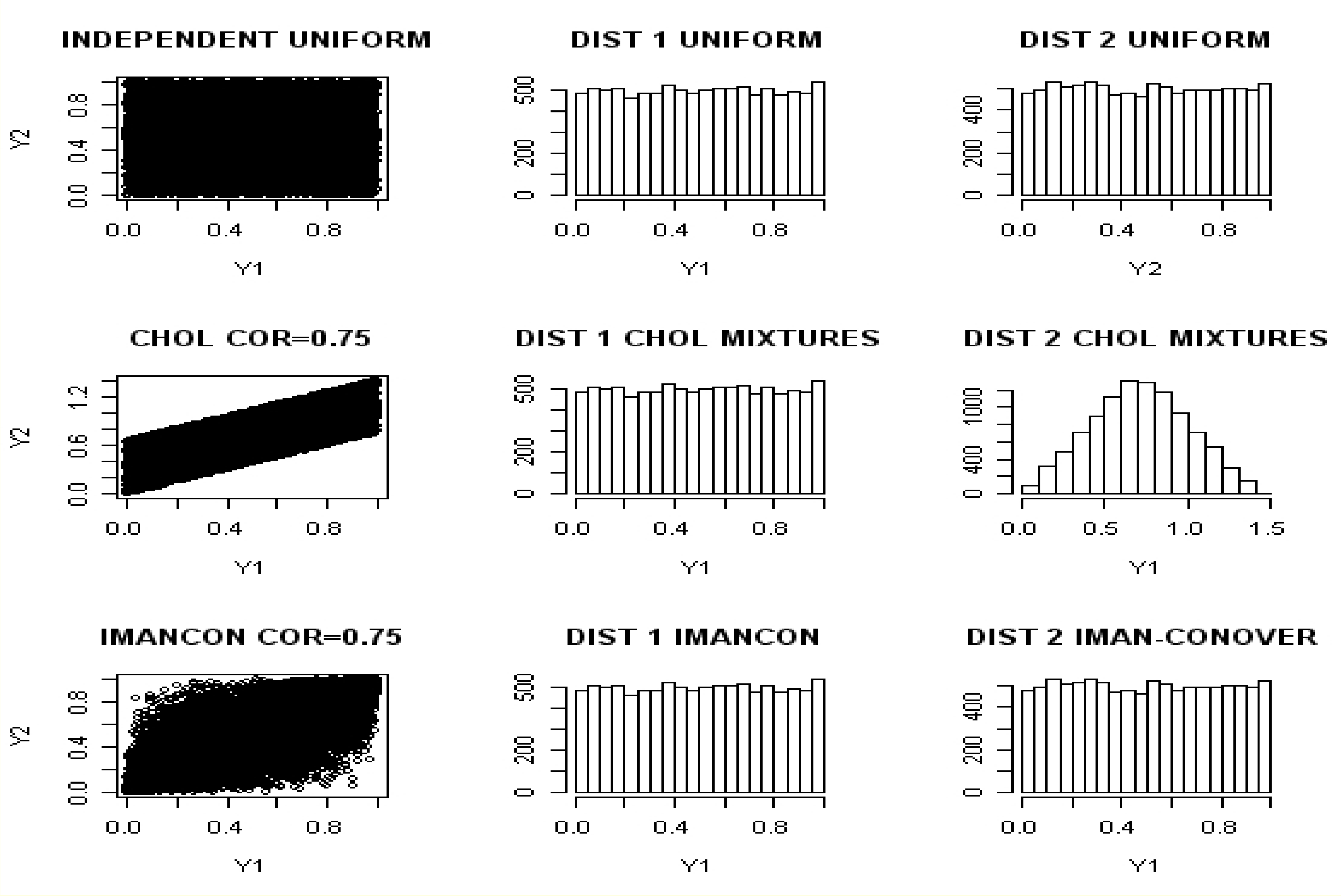
REINSURANCE COMPANY REQUIREMENT (cont.)

- Friday Call -- Drop Dead Date: Monday Morning
- Used the Iman-Conover Process
  - Preserves original marginal distributions on yields and quality
  - Introduces correlation between random variates
  - Equivalent to using Normal copula process described in upcoming process.
  - Used in @Risk software

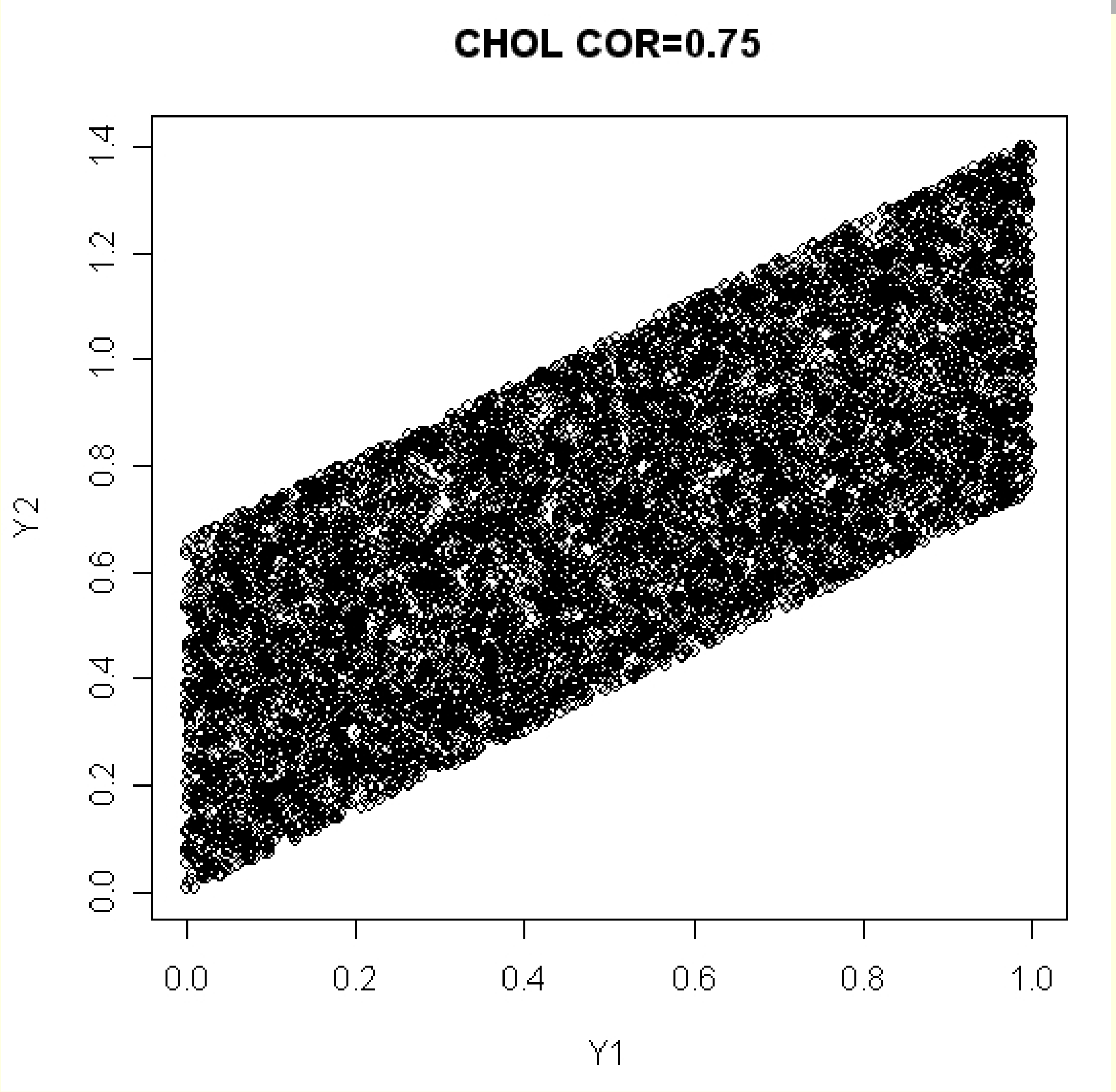
A VARIATION OF THE IMAN-CONOVER PROCESS

- Given Marginal Distributions
- Generate N x K independent sample Y_I
- Estimate or assume correlation structure
- Generate N x K multivariate Normal sample Z_C with correlation structure Σ
- Construct the correlated matrix Y_C by reordering the elements from each column in Y_I to have the same rank order as that of the corresponding column in Z_C.

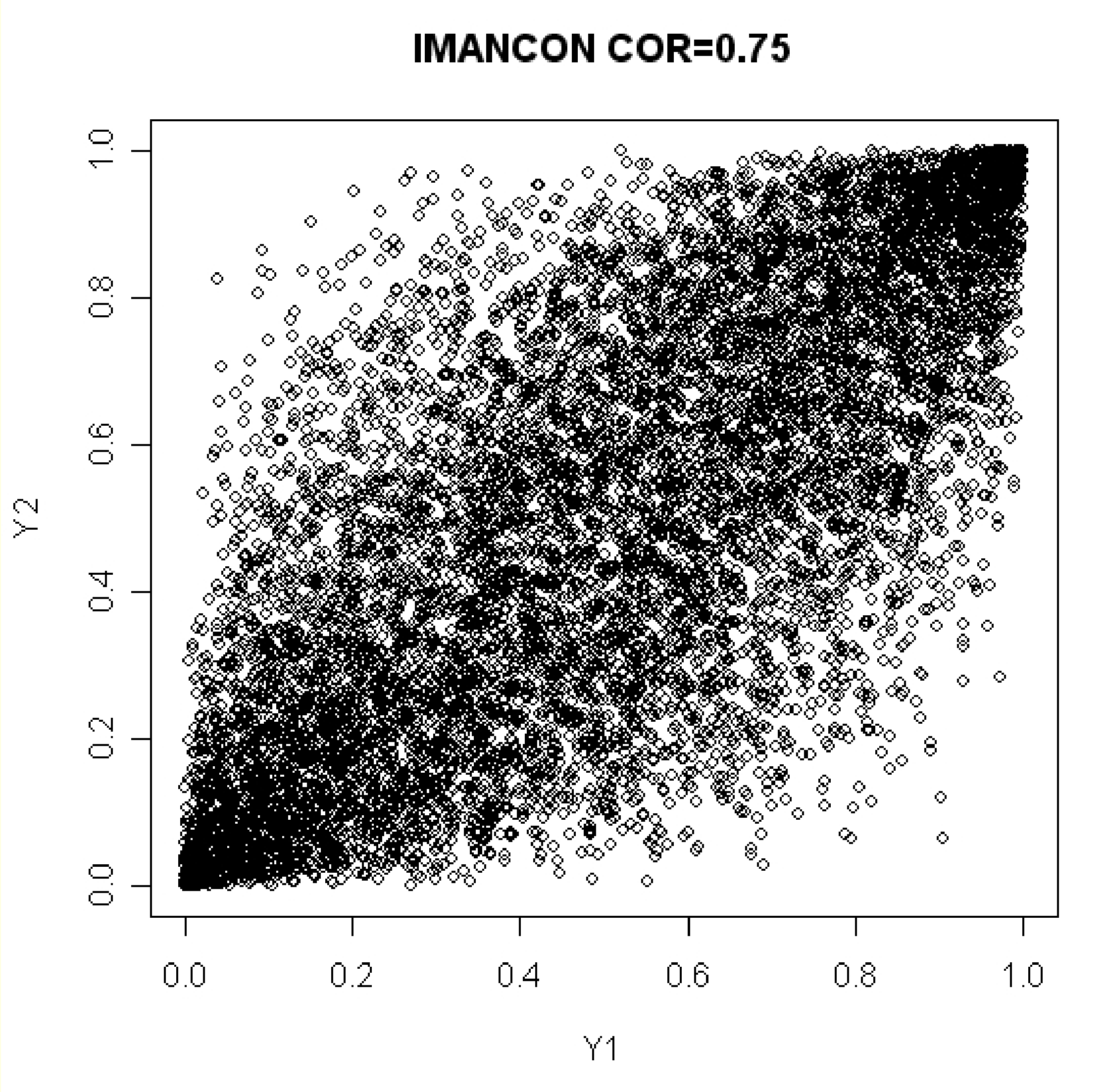
EXAMPLE WITH UNIFORM MARGINAL DISTRIBUTIONS



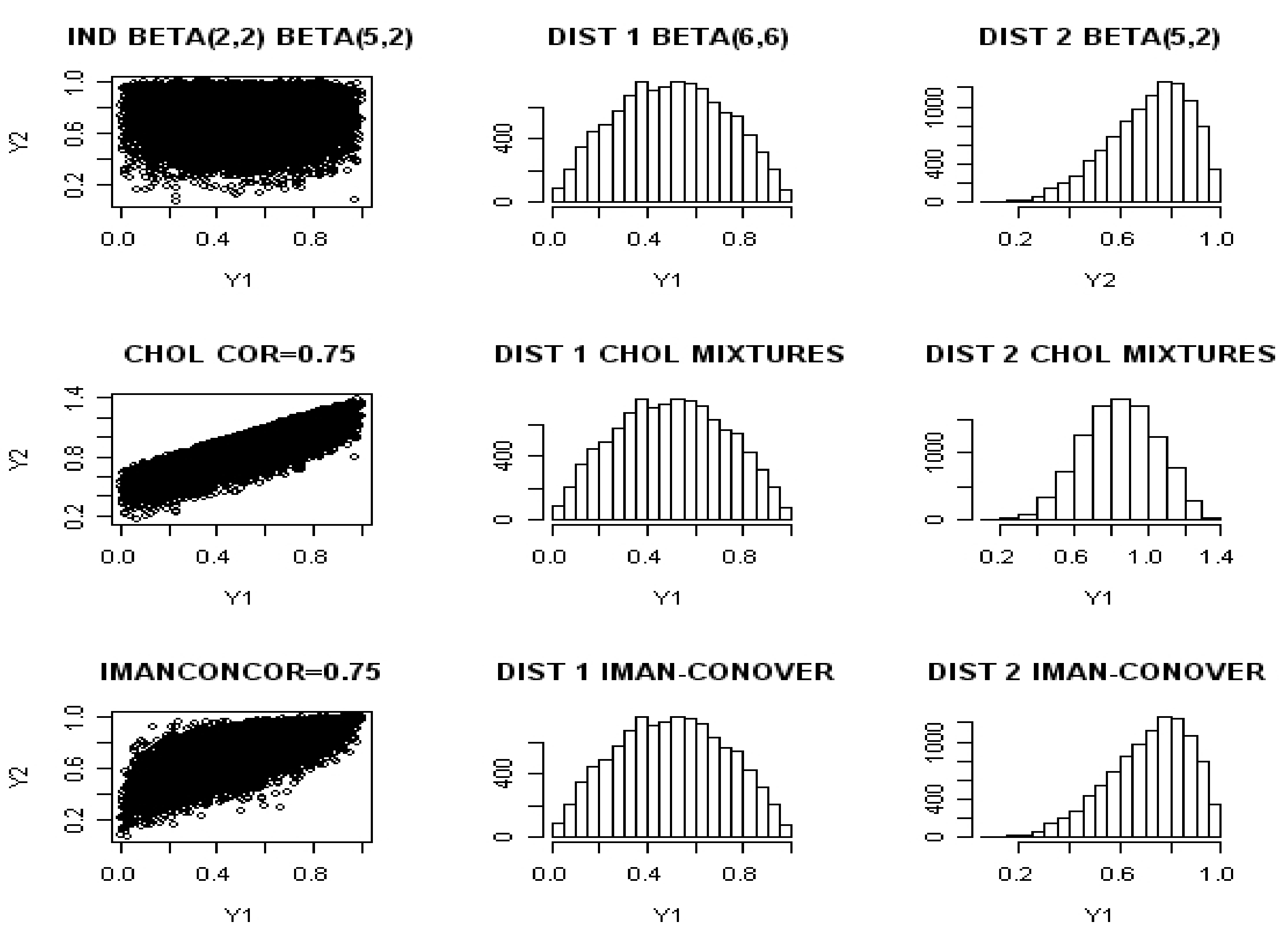
JOINT UNIFORM REALIZATIONS WHEN CORRELATION INTRODUCED BY APPLYING CHOLESKI FACTORIZATION DIRECTLY TO INDEPENDENT MARGINALS



IMAN-CONOVER JOINT UNIFORM REALIZATIONS



EXAMPLE WITH BETA MARGINAL DISTRIBUTIONS



REINSURANCE COMPANY (cont.)

- Completed analysis with estimated VAR levels for simulated book of business
- Procedures approved and the project accepted
- Iman-Conover procedure probably most widely used procedure for introducing dependencies between variates while preserving marginal distributions (Haas)

REINSURANCE COMPANY (cont.)

- Results equivalent to those generated using a special case of a more general method of modeling dependencies between random variables.
- The MV-Normal variant of the Iman-Conover process is equivalent to using the normal COPULA method
- Copulas are multivariate uniform distributions each with their own dependency structure
- (Nelsen; Cherubini et. al; McNeil et. al)

OVERVIEW OF SIMULATING DEPENDENCIES WITH COPULA METHODS

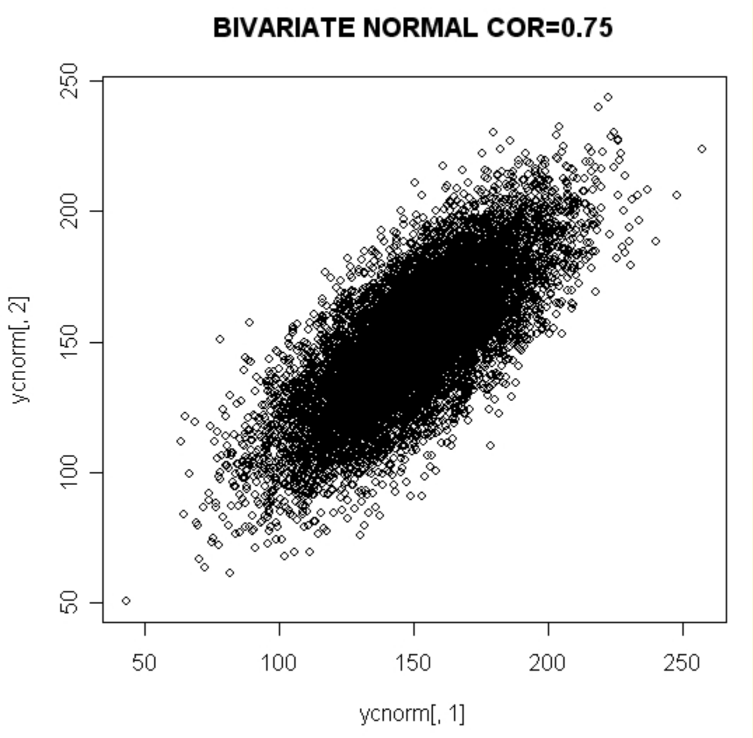
- Given Marginal Distributions
- Generate N x K independent sample Y_I using given marginals
- Estimate or assume dependence structure
- Generate N x K multivariate UNIFORM sample Z_C with desired dependence structure (the sample is generated by creating random samples from a Copula)

OVERVIEW OF SIMULATING DEPENDENCIES WITH COPULA METHODS (cont.)

- Construct the jointly dependent matrix Y_C by reordering the elements from each column in Y_I to have the same rank order as that of the corresponding column in Z_C
- Note that all characteristics of the marginal distributions in each column of Y_I are retained
- **A more detailed justification for this process is presented below**

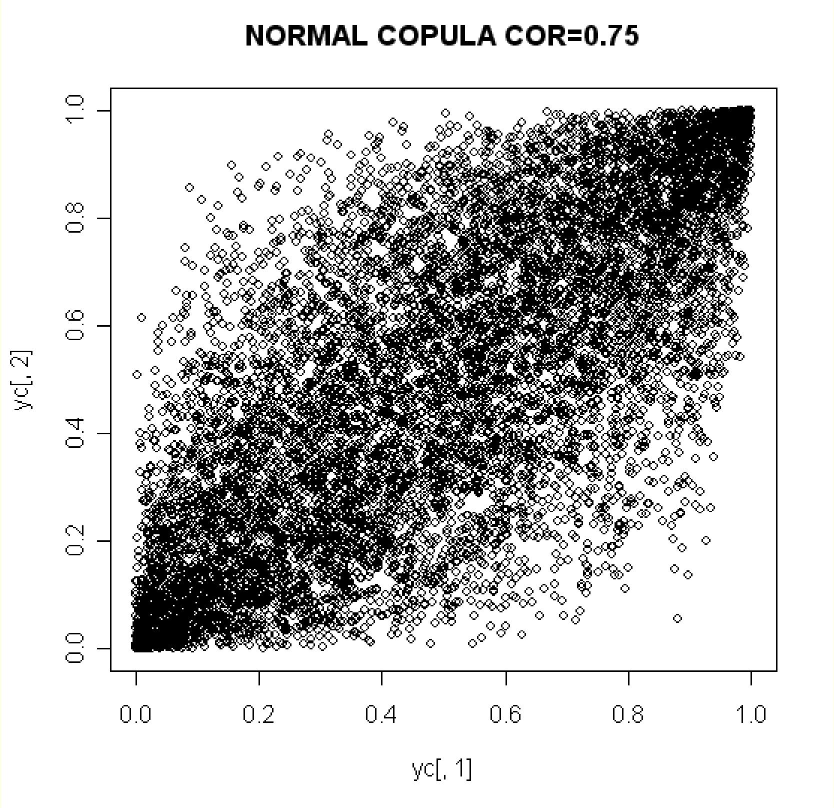
MOTIVATIONS FOR COPULA METHODS

- Iman-Conover (MV-Norm Variant) implicitly assumes elliptical covariate dependencies (Example: Margins Normal(150,25))



MOTIVATIONS FOR COPULA METHODS (cont.)

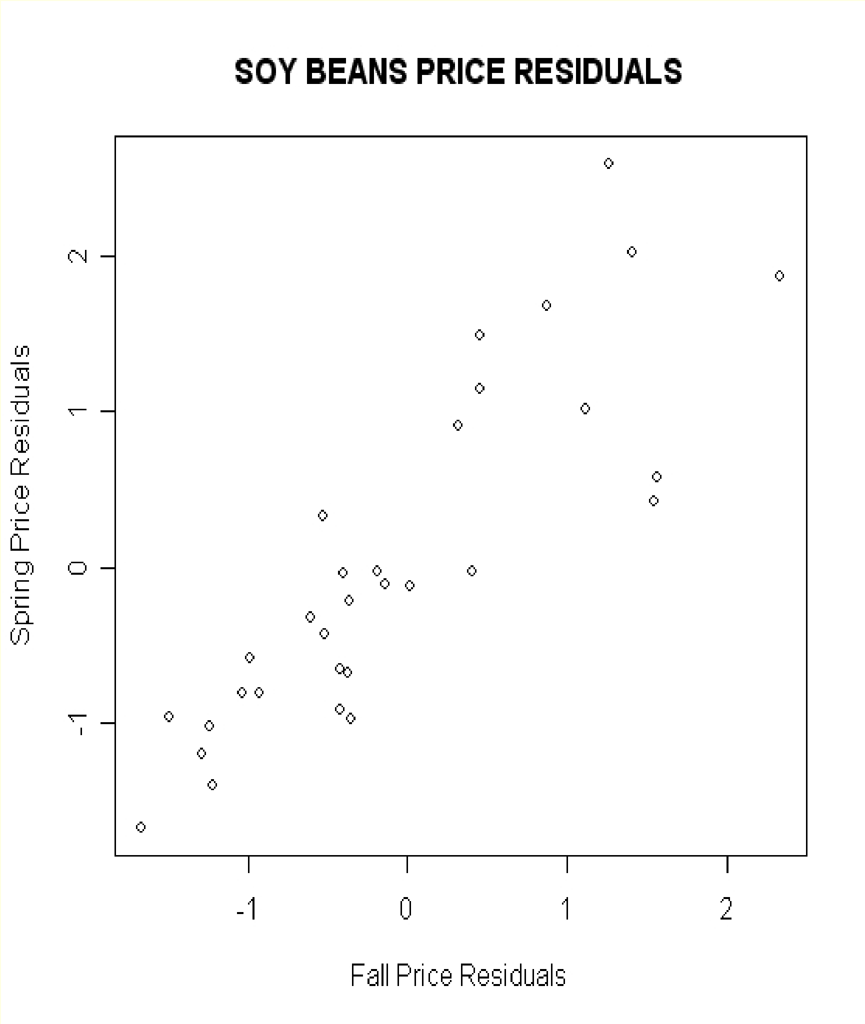
- The above bivariate normal sample was generated using the Copula sample:

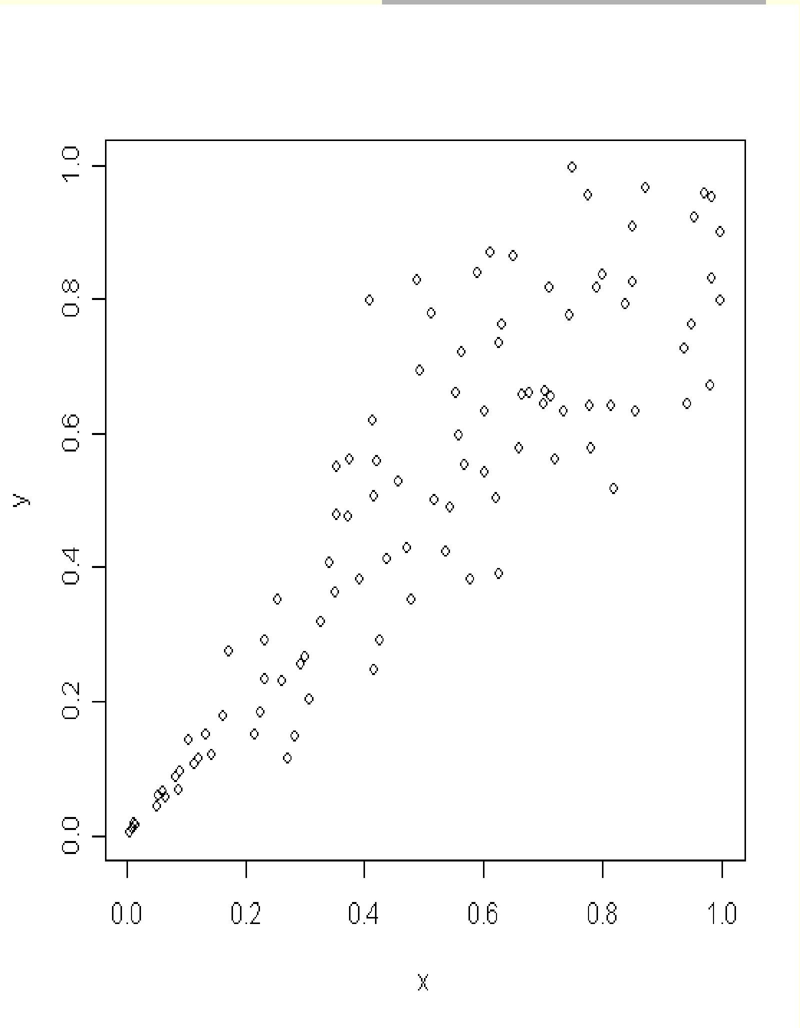


MOTIVATIONS FOR COPULA METHODS (cont.)

- Note the elliptical nature of the bivariate sample and the corresponding copula
- The copula realizations are multivariate uniform
- HOWEVER:

PLOTS OF FINANCIAL DATA OFTEN SHOW DIFFERENT RELATIONSHIPS





PLOTS OF FINANCIAL DATA OFTEN SHOW DIFFERENT RELATIONSHIPS (cont.)

- Financial data often exhibit asymmetric dependencies with “tighter” relationships during economic downturns and “looser” relationships during average or good economic times
- Asymmetric dependencies can be modeled with multivariate uniform distributions (Copulas)

COPULA DEFINITIONS AND RESULTS

COPULA: **"A d-dimensional copula is a distribution function on $[0,1]^d$ with standard uniform marginal distributions" (McNeil et al.)**

- A copula C(**u**) : $[0,1]^d \rightarrow [0,1]$ is a function that maps the d-dimensional unit hypercube into the unit interval **(**McNeil et al.)
- To qualify as a copula (or an d-dimensional distribution function), the copula

  C(**u**) : $[0,1]^d \rightarrow [0,1]$ must satisfy three conditions discussed by Nelsen pp 37-44. This discussion is beyond the scope of this paper

Sklar's Theorem

(Nelsen p 41)

- **Key Result:**
  Let H be any n-dimensional distribution with marginal distributions F_1, F_2, …F_n. Then there exists an n-copula C such that for all x in $\overline{\mathrm{R}}^n$

$$H(x_1, x_{2,} \cdots, x_n) = C(F_1(x_1), F_2(x_2), \cdots F_n(x_n))$$

If all F_i are all continuous then C is unique …….
Conversely if C is an n-copula and F_1, F_2, …F_n are distribution functions, H as defined above is an n-dimensional distribution function with margins F_1, F_2, …F_n.

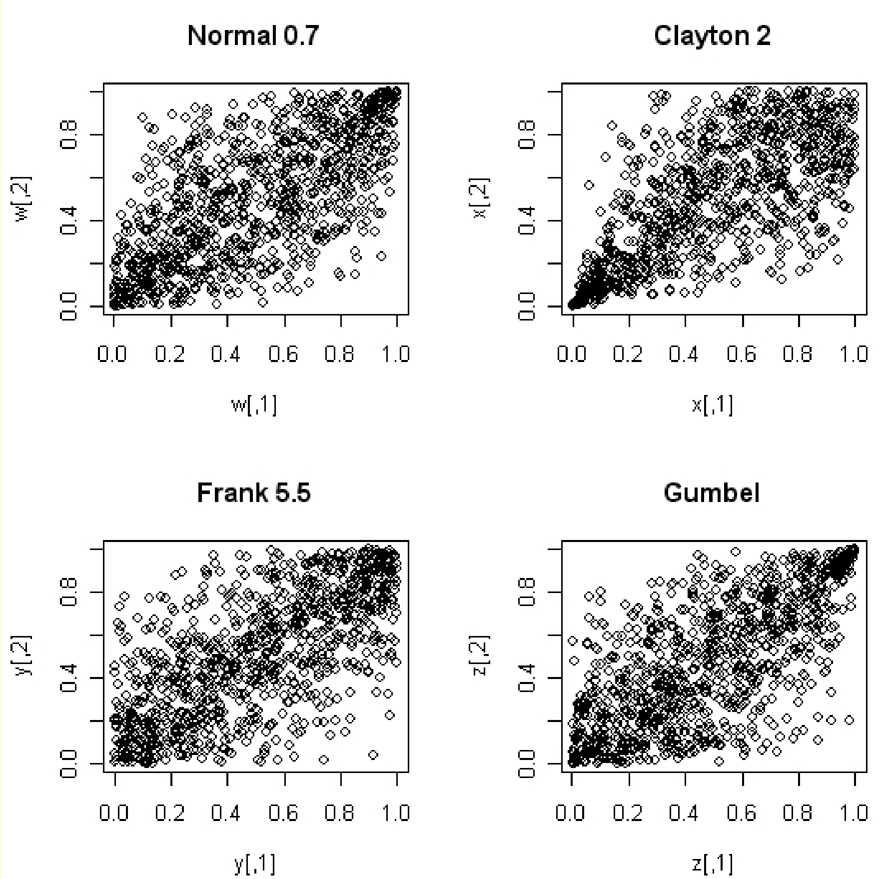
References: Nelsen; Chiappori, Luciano, Vecchiato; McNeil, Frey, Embrechts

Sklar's Theorem (cont.) (Nelsen p 41)

- This result allows us to simulate joint distributions with a two step process.
  - Estimation of appropriate marginal distributions (not necessarily from the same family)
  - Estimate or assume an appropriate copula.

EXAMPLES OF COMMONLY USED COPULAS

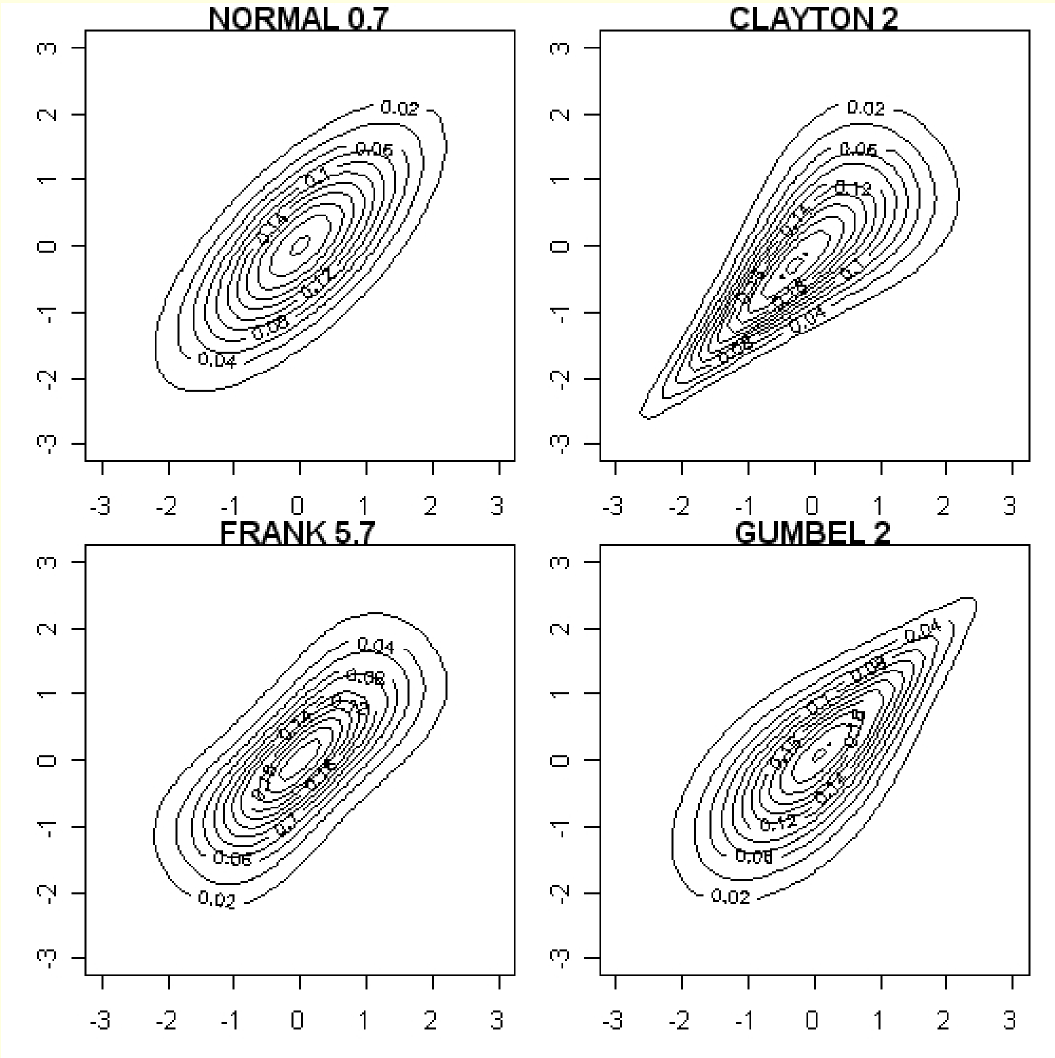
(GENERATED WITH JUN YAN'S COPULA PACKAGE FOR R)



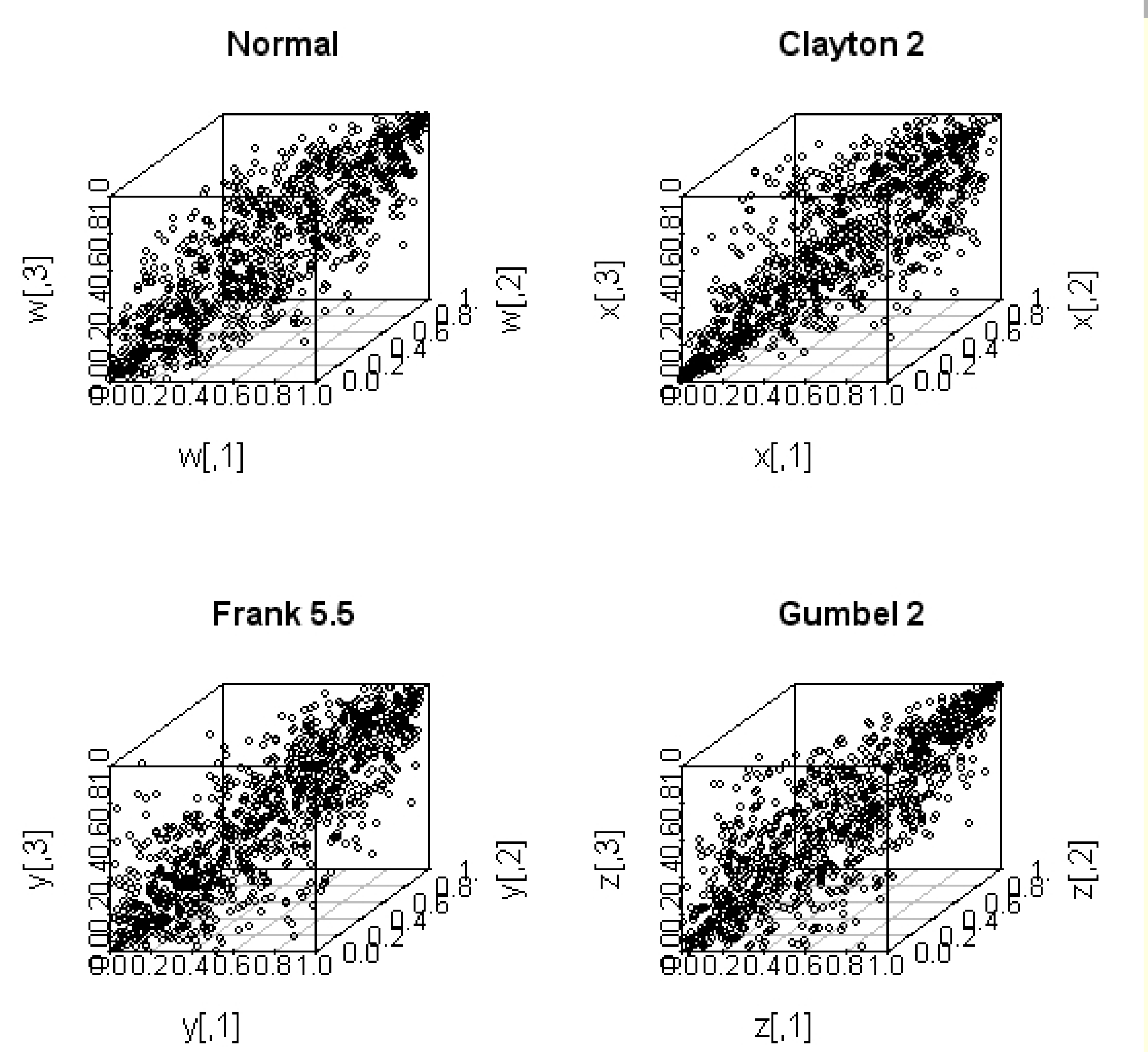
- Recall that these are joint Copula realizations i.e. joint uniform variate draws and are thus defined in the $[0,1]^2$ space.

LEVEL CURVES WITH NORMAL(0,1) MARGINALS

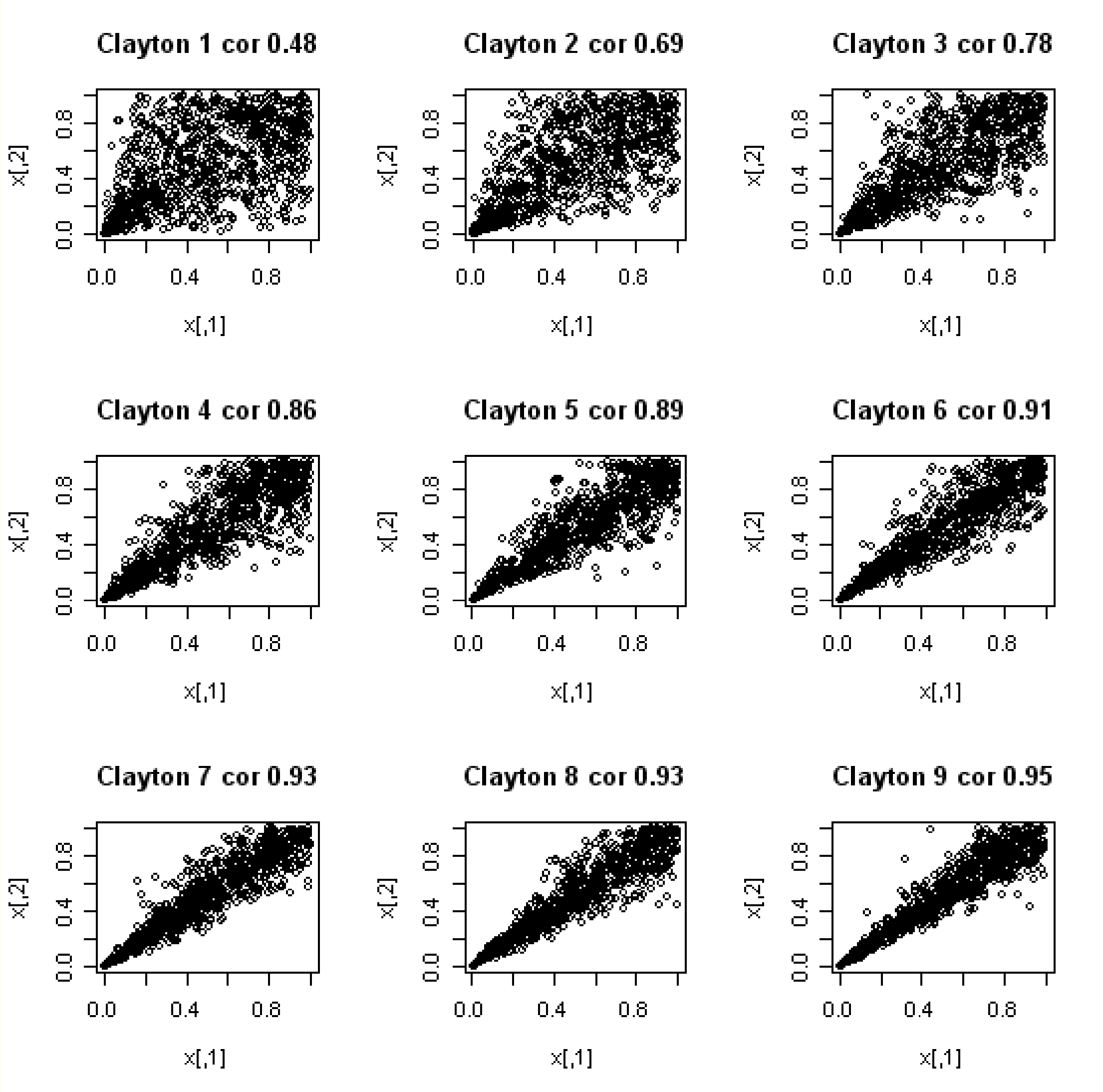
AND VARYING COPULAS (Jun Wan-R)



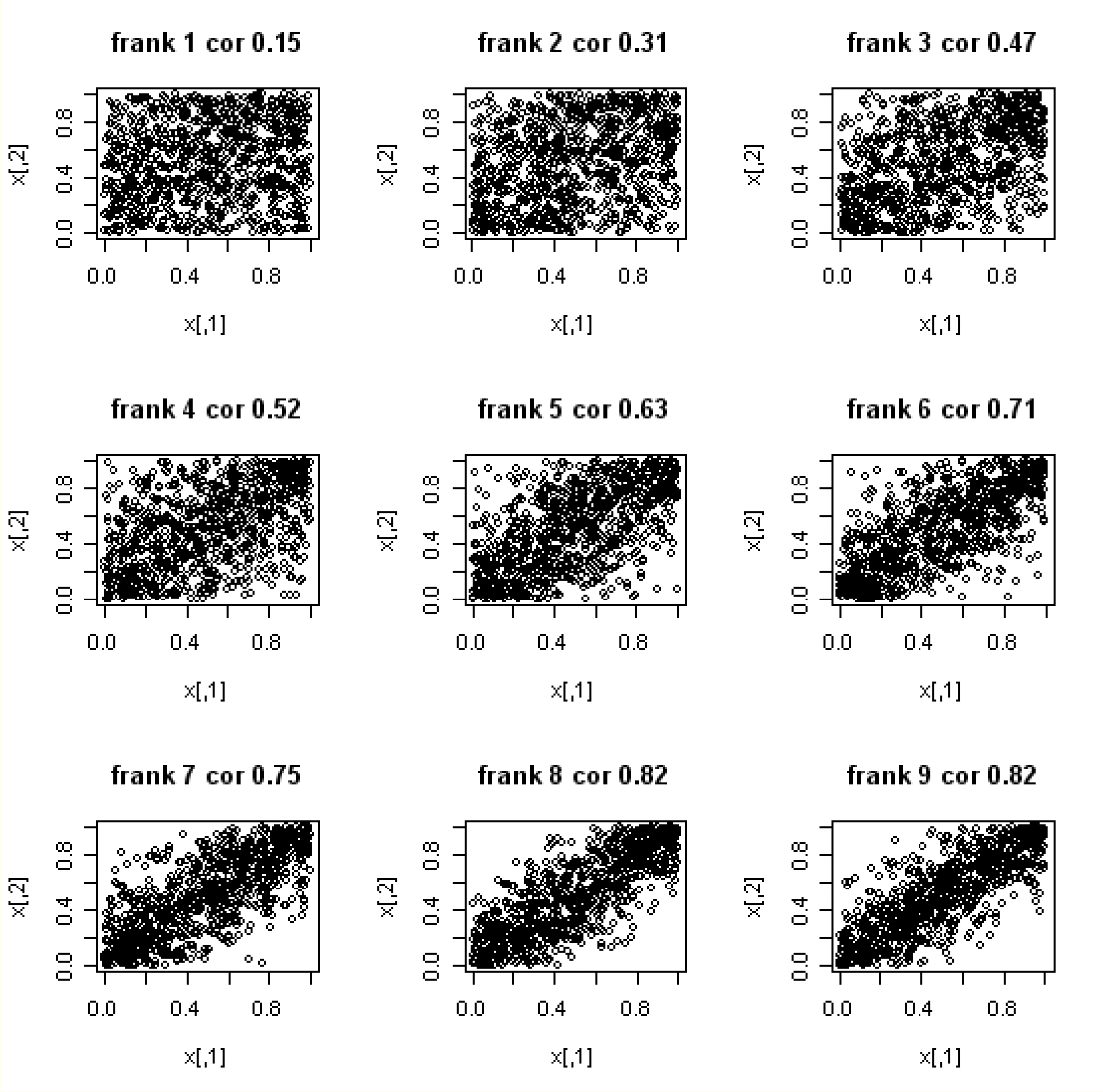
THREE DIMENSIONAL COPULA SCATTER PLOTS



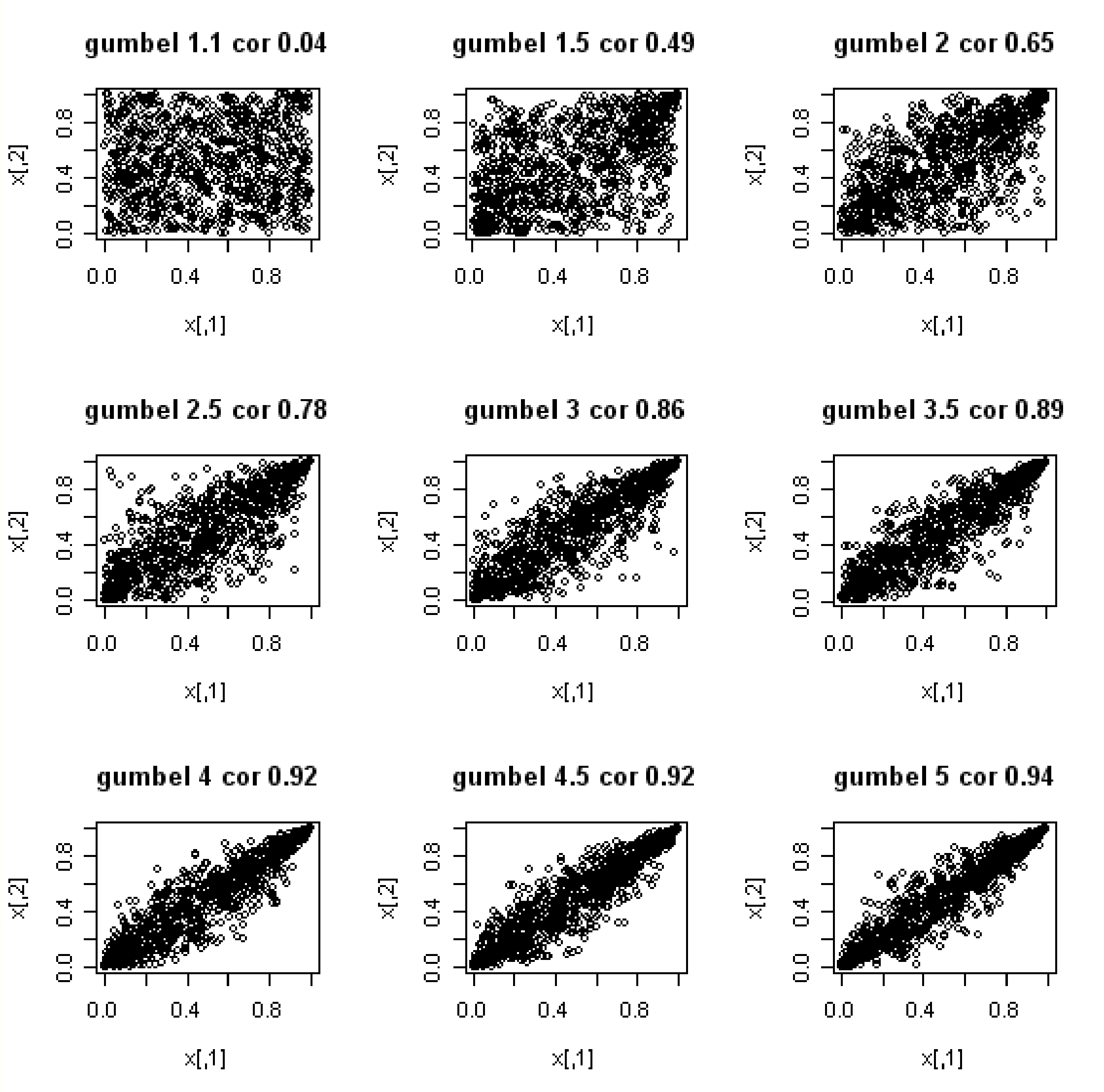
SCATTER PLOTS FROM CLAYTON COPULAS



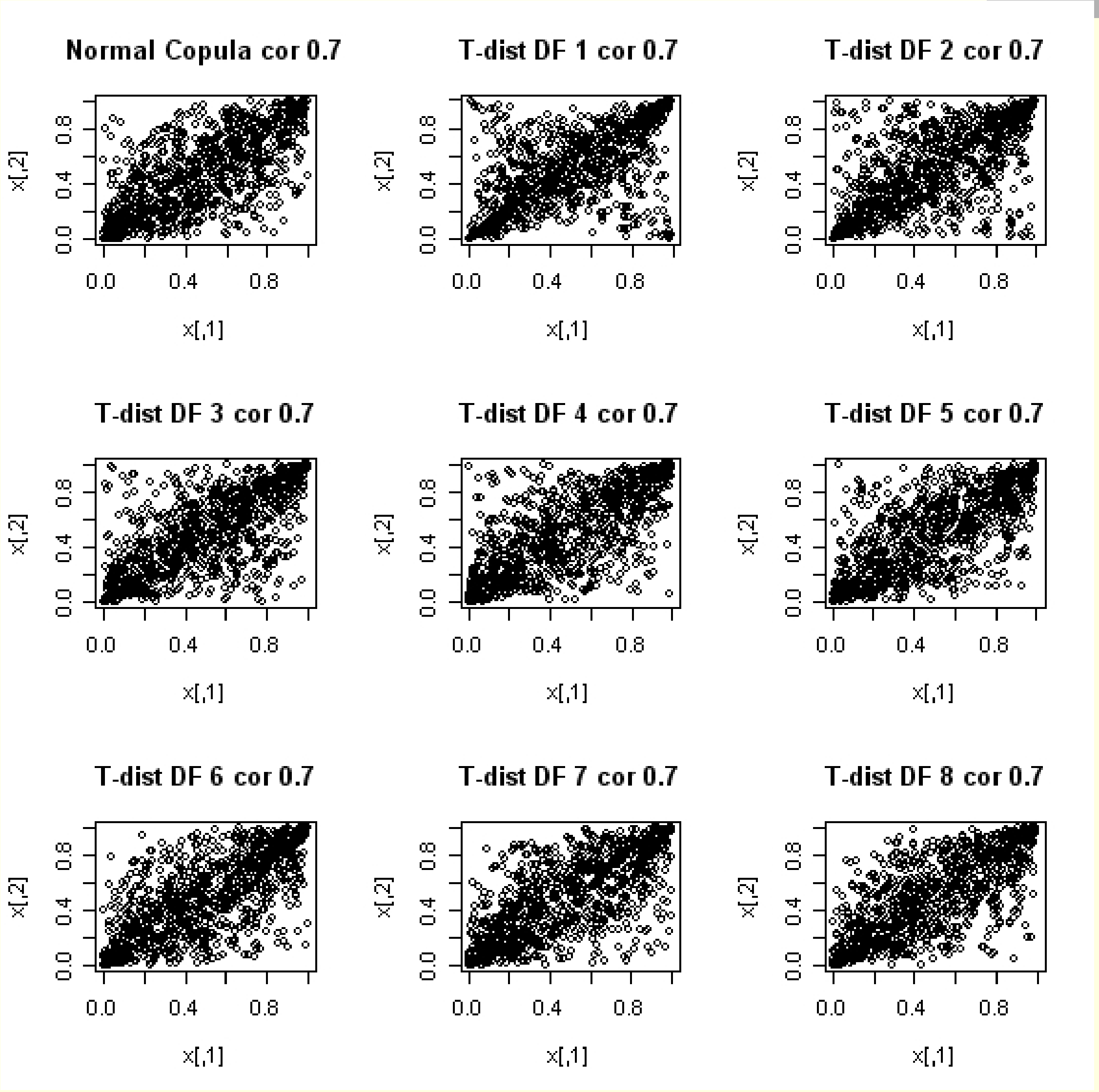
SCATTER PLOTS FROM FRANK COPULAS



SCATTER PLOTS FROM GUMBEL COPULAS



SCATTER PLOTS FROM T-COPULAS



EXAMPLES

- ESTIMATING ENTERPRISE LEVEL DISCOUNTS

- ESTIMATION OF VALUE AT RISK FOR BOOK OF BUSINESS

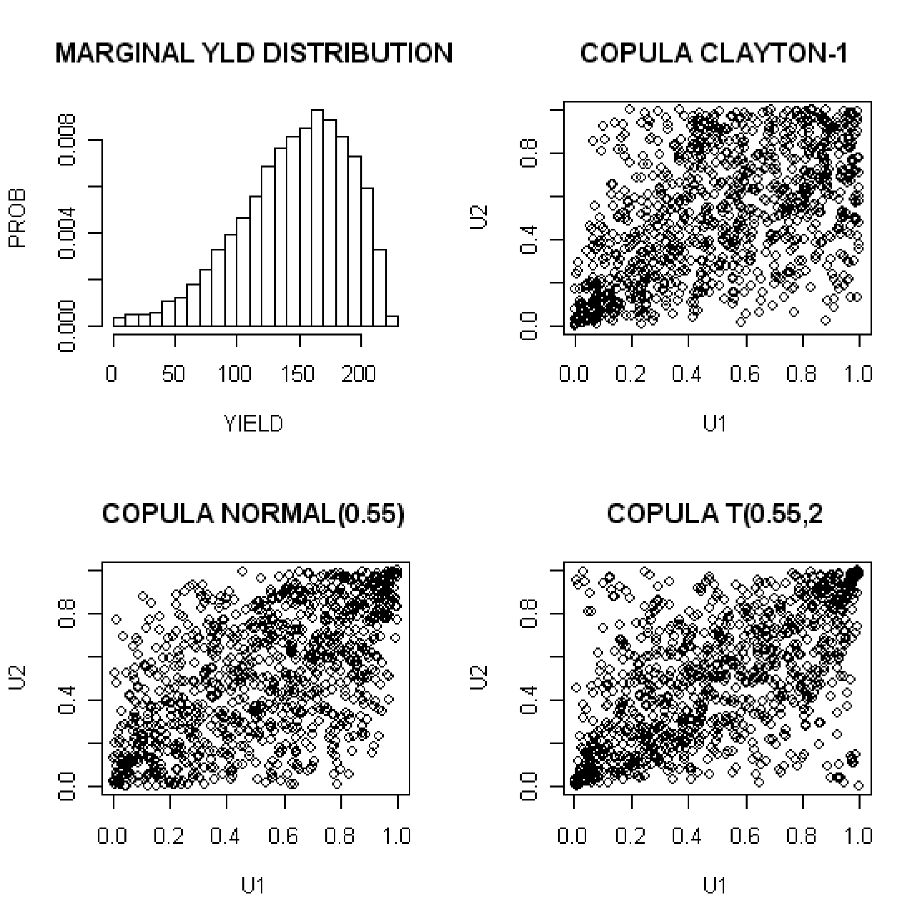
ASSUMPTIONS FOR EXAMPLES

- MARGINAL BASE FARM YIELDS DISTRIBUTED BETA(4, 2, 0, 225)
  - LEFT SKEWED
  - MEAN = 150   SD = 40
- 5% PROBABILITY OF HAIL EVENT
  - Given hail event proportional losses distributed UNIF(0,1)

EXAMPLE: (Cont.)

- GENERATED K INDEPENDENT MARGINALS SAMPLE OF SIZE 10000
- GENERATED 10000 BY K JOINT SAMPLE BY APPLYING COPULAS
  - CLAYTON-1
  - NORMAL (COR=0.55)
  - T(COR=0.55, DF=2)

EXAMPLE: (Cont.)



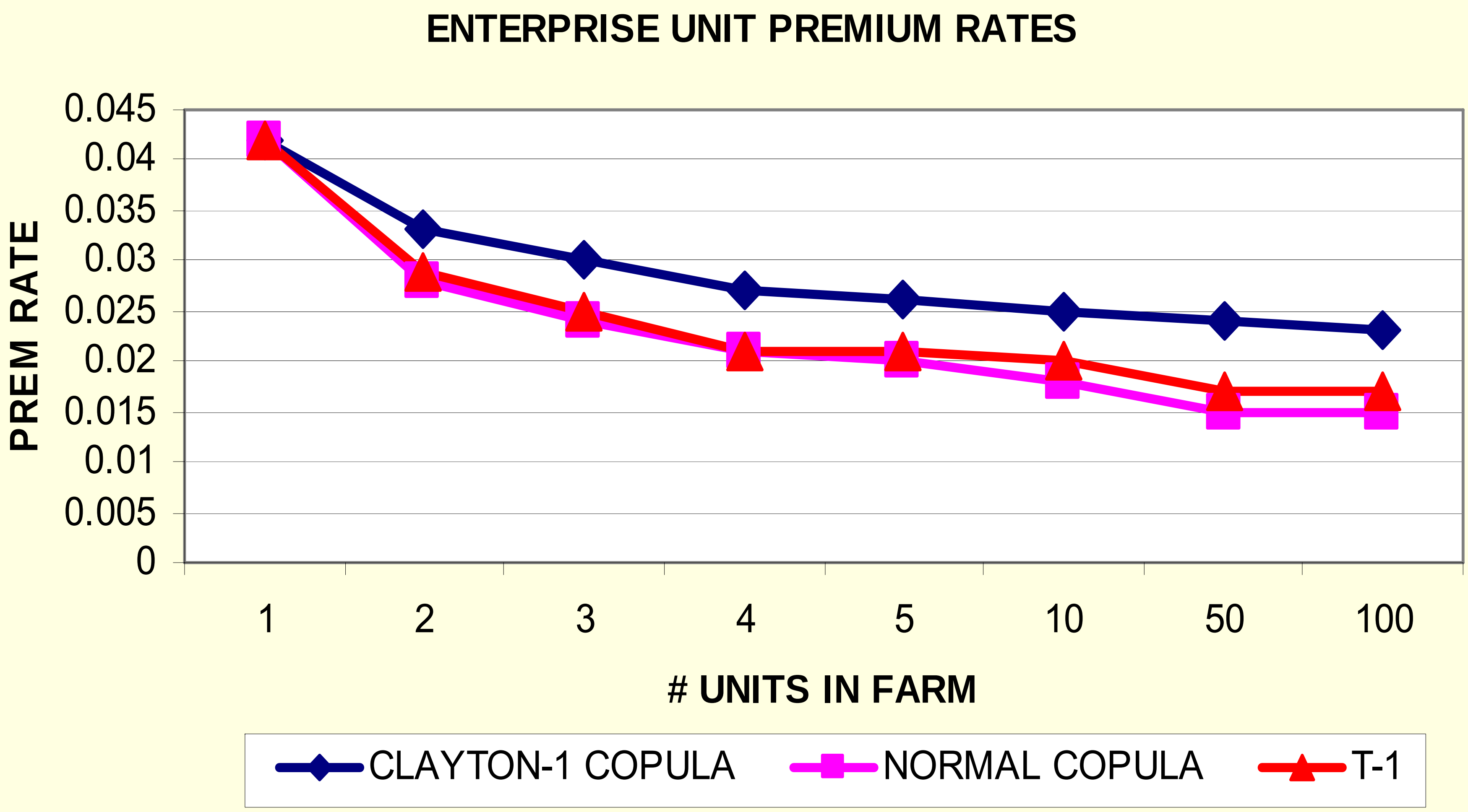
EXAMPLE: (Cont.)

- COMPUTED ENTERPRISE UNIT YIELDS AS AVERAGE YIELDS ACROSS THE K “UNITS” FOR

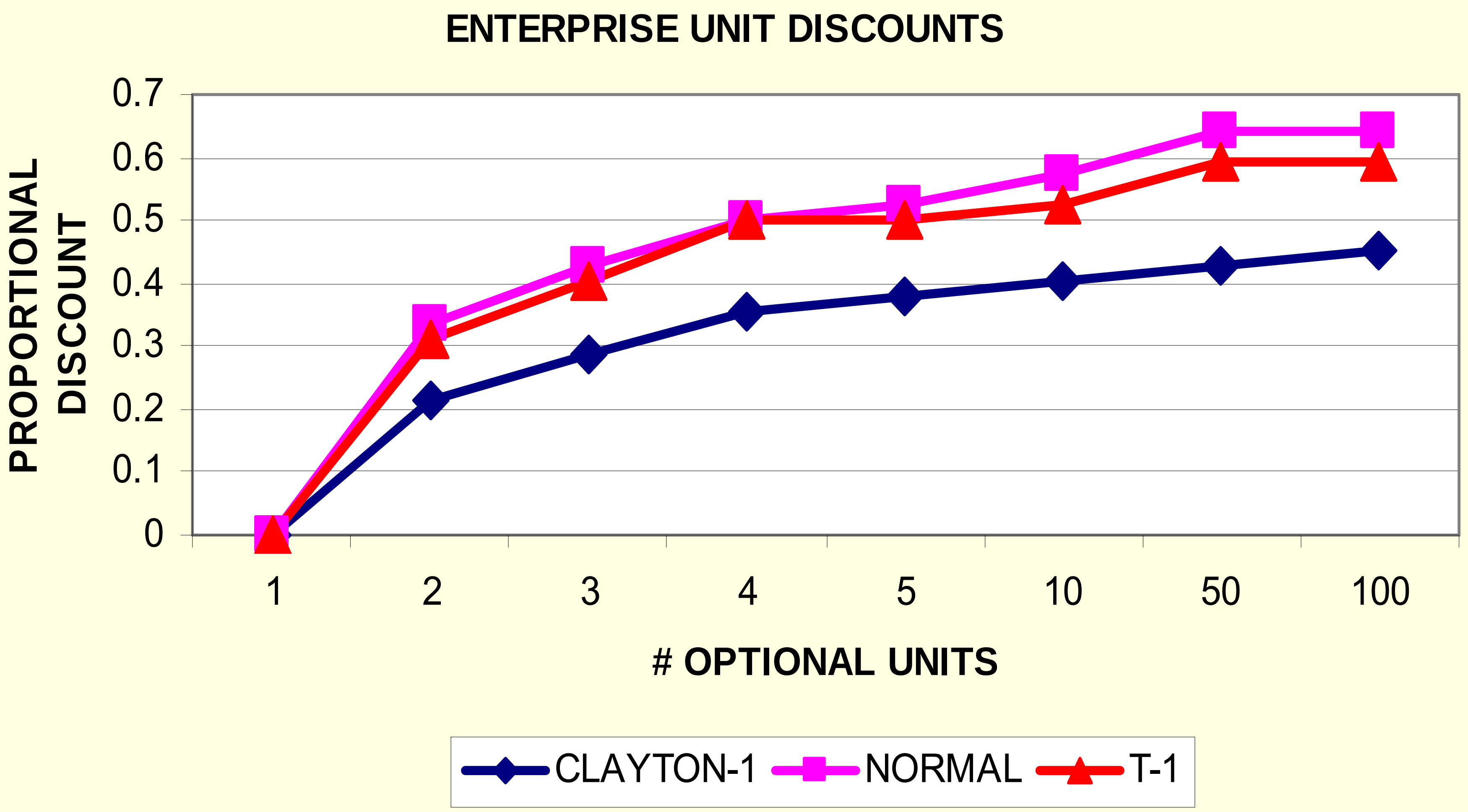
  K = 2, …, 100 UNITS

  - COMPUTED 65 % CVG INDEMNITIES FOR “ENTERPRISE” UNIT
  - COMPUTED AVERAGE LCR
    - COMPUTED 65 % INDEMNITIES ON EACH “OPTIONAL UNIT”
  - AGGREGATED INDEMNITIES ACROSS “OPTIONAL UNITS”
  - COMPUTED 1% AND 5% VAR ON A PER ACRE BASIS

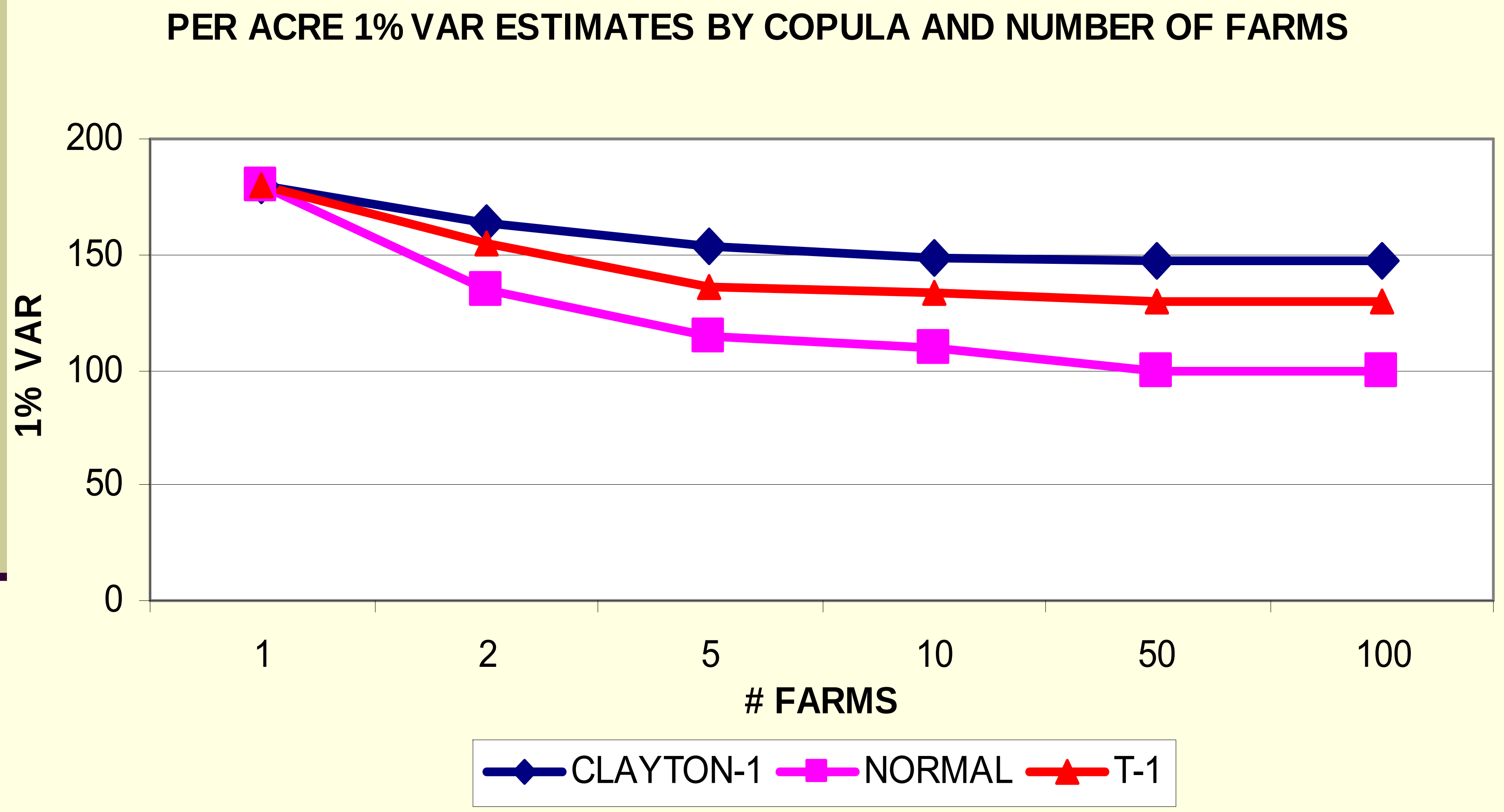
APPLICATIONS
ENTERPRISE UNIT DISCOUNT EXAMPLE



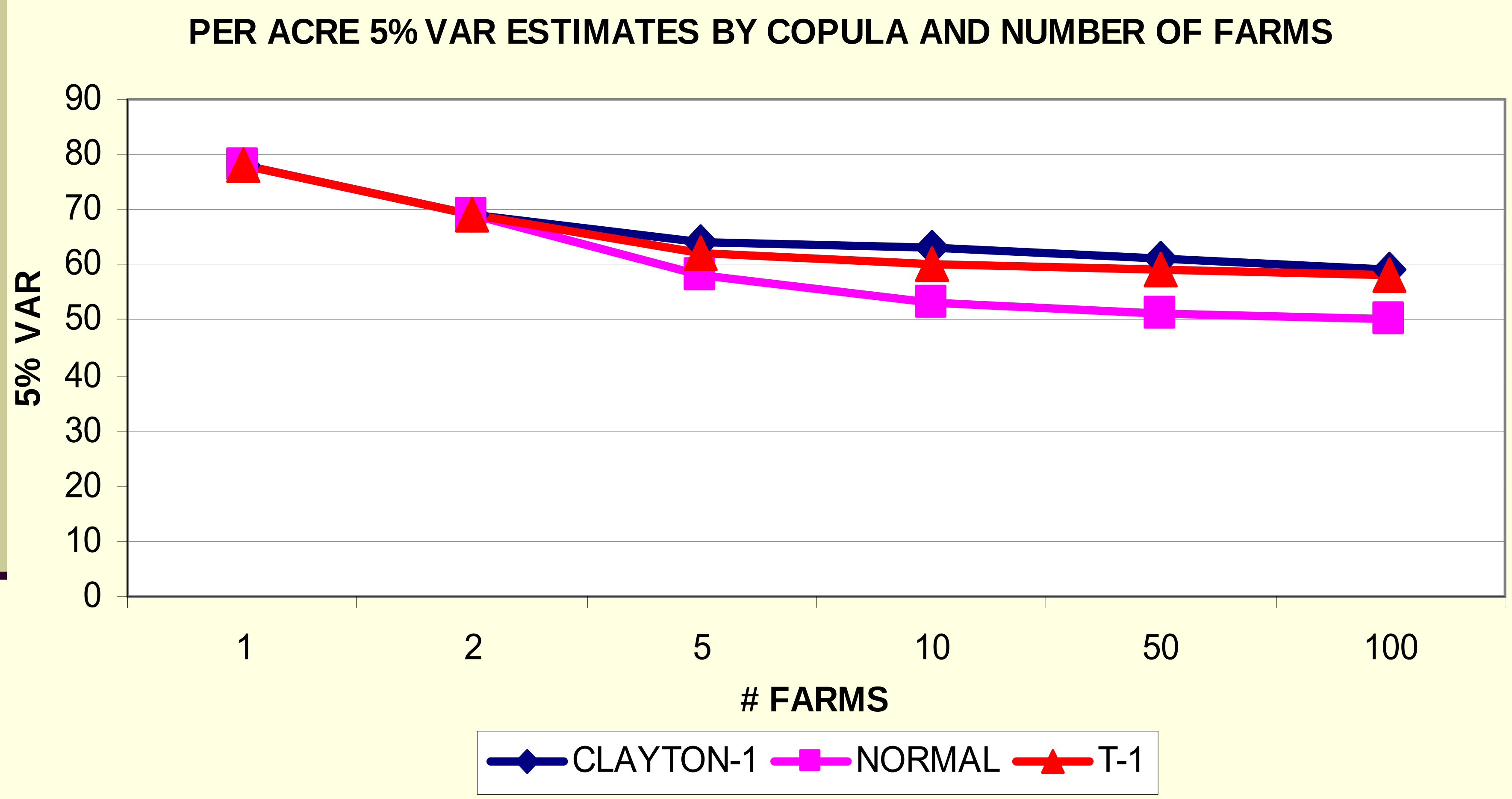
APPLICATIONS ENTERPRISE UNIT DISCOUNT EXAMPLE (cont.)



ONE PERCENT VAR ESTIMATES



FIVE PERCENT VAR ESTIMATES



LIMITATIONS

Selecting Appropriate Copula

(An Infinite Number Exist)

- Empirical Copula
- Nonparametric kernel smoothing methods (Cherubini et al.)
- Maximum likelihood (Jun Yan's R package)

LIMITATIONS (cont.)

Limited Ability To Model Different Dependency Relationships Between Different Marginals

- Currently normal or t-copulas most utilized if different “correlations” desired between different marginals
- Current versions of Archimedean Copulas (Clayton, Frank, Gumbel) are quite restrictive with respect to allowing heterogeneous dependency structures in higher dimensions
- Work continues in this area

CONCLUSIONS

- Increasing use of market basket (RA or LGM) and/or other index type insurance or marketing products
- Appropriate rates and prices of market basket/index products can be different under different Copula structures
- Examining the effects of different Copula structures in n-dimensions facilitated by freely available software such as Jun Yan’s Copula package for R
- Copulas are becoming increasingly used in the finance and insurance industry and are a valuable tool for the applied researcher