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Francesco Menoncin, Sergio Vergalli

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By Francesco Menoncin, Università degli Studi di Brescia
Sergio Vergalli, Università degli Studi di Brescia

Summary

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Keywords: Optimal Stopping Time, Retirement Choice, Labour Supply, Asset Allocation, Mortality Risk

JEL Classification: C61, D15, G11, J22

Address for correspondence:
Sergio Vergalli
Università degli Studi di Brescia
Department of Economic and Management
Via S. Faustino 74/B
25122 Brescia
Italy
E-mail: sergio.vergalli@unibs.it

Optimal stopping time, consumption, labour, and portfolio decision for a pension scheme

Francesco Menoncin*and Sergio Vergalli†

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Abstract

In this work we solve in a closed form the problem of an agent who wants to optimise the inter-temporal utility of both his consumption and leisure by choosing: (i) the optimal inter-temporal consumption, (ii) the optimal inter-temporal labour supply, (iii) the optimal share of wealth to invest in a risky asset, and (iv) the optimal retirement age. The wage of the agent is assumed to be stochastic and correlated with the risky asset on the financial market. The problem is split into two sub-problems: the optimal consumption, labour, and portfolio problem is solved first, and then the optimal stopping time is approached. The martingale method is used for the first problem, and it allows to solve it for any value of the stopping time which is just considered as a stochastic variable. The problem of the agent is solved by assuming that after retirement he received a utility that is proportional to the remaining human capital. Finally, a numerical simulation is presented for showing the behaviour over time of the optimal solution.

1 Introduction

According to the Finnish Centre for Pensions (http://www.etk.fi/en), in the EU15 states, the average retirement age is 65 years. Instead, in the other 12 EU countries, the retirement age is lower, but it is planned to be raised to the same level over the next decade. Furthermore, Germany, Denmark, France, and Spain have decided to increase the retirement age from 65 to 67 years, while the goal is 68 years in UK and Ireland.

Most of the changes in retirement ages are scheduled for the 2020s. In Italy, the Netherlands, Finland, Cyprus, Denmark, Greece, Portugal, UK and Slovakia, the retirement age will be linked to the development of the expected life expectancy.

*francesco.menoncin@unibs.it – Department of Economic and Management, Università degli Studi di Brescia, Via S. Faustino 74/B – 25122 Brescia – Italy

†sergio.vergalli@unibs.it – Department of Economic and Management, Università degli Studi di Brescia, Via S. Faustino 74/B – 25122 Brescia – Italy

"European pension systems are facing the dual challenge of remaining financially sustainable and being able to provide Europeans with an adequate income in retirement" (European Commission, 2017). Both the recent and the scheduled increases in the retirement age answer precisely the first goal to keep the pension systems financially sustainable.

Actually, life expectancy after the age of 70 has been constantly increasing over the last decades, generating the so-called "longevity risk" (European Commission, 2018b). Thus, without a counterbalance increase in the retirement age, the level of contributions paid by the workers, could not be able to suitably finance the pensions that will have to be paid for a longer and longer period of time.

It is precisely because of this increasing longevity risk, that "the defined contribution (DC) pension schemes are becoming more and more important in the pension system and are replacing the defined benefit (DB) that were more frequent in the past" (Francesco Menoncin and Elena Vigna, 2017)

The optimal investment strategy in the accumulation phase (i.e. prior to retirement) in a DC framework has been derived in the literature with a variety of objective functions and financial market structures, see, among many others, Boulier, S. J. Huang and Taillard (2001), Haberman and E. Vigna (2001), Deelstra, Grasselli and Koehl (2003), Devolder, Bosch Princep and Dominguez Fabian (2003), Battocchio and F. Menoncin (2004), Cairns, D. Blake and Dowd (2006) and Di Giacinto, Federico and Gozzi (2011). Nevertheless, setting a statutory retirement target that is able to guarantee both the time financial sustainability and an adequate income in old ages (that is macro and microeconomic targets) is a very demanding task and it could result in an exogenous target that may not be perfectly in line with a worker optimal choice.

Actually, in most countries the average effective age at which older workers withdraw from the labour force (the so-called "effective age of retirement") is well below the level that could allow to finance a full old-age pension.[1]

The European services, EPC, show that "for men, 17 countries show effective retirement ages that are on average 0.9 year lower than the estimated labour exit ages. This difference amounts to more than 1.5 years for Bulgaria, the Czech Republic, Portugal and Romania. In six countries, men enter the pension system at a later age than they leave the labour market, with an average gap of 1 year. Only for the United Kingdom, the difference exceeds 1.5 years" (European Commission, 2018b).

A market solution for managing the longevity risk is the issuance of longevity-linked assets that could counterbalance the reduction of income after retirement.

Unfortunately, the market for such assets remain quite illiquid. In fact, a branch of the literature has analysed this market, looking for the reasons that may have contributed to undermine the market development, such as: (i) the lack of standardisation, (ii) informational asymmetries, and (iii) basis risk.

Literature has recently modelled the systematic randomness in mortality (e.g. Lee and Carter, 1992), the design and evaluation of hedging instruments

[1] http://www.oecd.org/els/emp/average-effective-age-of-retirement.htm

(A. Blake D. C., Dowd and MacMinn, 2006; Denuit, Devolder and Goderniaux, 2007), and the management of longevity risk (Barrieu et al., 2012). Recently, Francesco Menoncin and Regis (2017) found that individuals should optimally invest a large fraction of their wealth in longevity-linked assets in the pre-retirement phase, because of their need to hedge against stochastic fluctuations in their remaining lifetime at retirement. Another manner to study the same problem is to calculate, for each individual, the optimal timing of retirement and understand how far it is from the exogenous target, taking into account some budget and labour constraints, and the uncertainty of the market.

In line with the previous statements, in this paper we solve the problem of a representative agent who must decide, at the same time: (i) how much to work, (ii) how much to invest on financial market, (iii) how much to consume, and (iv) when to retire. The agent's time horizon coincides with his death time and the optimisation problem is subject to some risks: (i) the market risk on the financial market (mainly price risk), (ii) the mortality risk on his lifetime, and (iii) the wage risk on his yield.

An extensive literature has explored consumption and investment decisions when mortality contingent claims are present. In particular, H. Huang and Milevsky (2008) analyse the decisions of households in the presence of income risk and life insurance. Explicit solutions are also obtained by Pirvu and Zhang (2012) with stochastic asset prices and inflation risk, and by Kwak and Lim (2014) with constant relative risk aversion (CRRA) preferences. All these papers consider a deterministic force of mortality, while in Francesco Menoncin and Regis (2017) it is modelled as a stochastic process.

We model the risk through a Wiener process and we solve the problem to maximise the expected utility of agent's inter-temporal consumption and leisure (non-working hours). After retirement, we assume the agent does not receive the wage any longer, while he still has to finance consumption. Nevertheless, at retirement he receives a utility which is proportional to his residual human capital. Here, we define the human capital as the total expected discount value of future wages.

This work builds on the same framework studied in F. Menoncin (2008) and Francesco Menoncin and Regis (2017). In both these papers, a longevity-linked security is listed on a complete arbitrage-free financial market. F. Menoncin (2008) solves the problem of a representative agent that must finance his consumption over his whole lifetime, without any labour income or pension scheme. Instead, Francesco Menoncin and Regis (2017) consider an agent that wants to maximise the utility of his inter-temporal consumption until the exogenously fixed retirement age. At retirement, the agent wants to accumulate enough wealth for financing the pension during his remaining lifetime. Again, in this framework no labour income is considered.

In this paper we solve the problem of an agent that has a labour income and must finance his inter-temporal consumption through the risky asset returns and this labour income during his working lifetime, while he has just to rely on the financial returns after retirement, when the other income is no more available. Furthermore, the agent can optimally choice when to retire.

We solve the problem in two steps: first we find the optimal consumption, labour supply, and portfolio for any possible (stochastic) stopping time, then we compute the optimal stopping time that maximises the value function of the first problem. This separability of the problem is achieved by solving it through the so-called martingale approach.

The framework of an optimal stopping time problem is often applied to the real options,[2] when an irrevocable decision must be taken. Actually, also in this case, the decision to retire cannot be called off and it permanently affects the dynamics of agent's problem. Bodie, Merton and Samuelson (1992) suggested that "one current research objective is to analyse the retirement problem as an optimal stopping problem and to evaluate the accompanying portfolio effects." Also Kula (2003) wrote that "the retirement decision may be treated as an investment process: first we collect capital / retirement wealth, and if we have accumulated enough we invest / retire, depending on the actual and expected prices / wages". Finally, Farhi and Panageas (2007) extended Bodie, Merton and Samuelson (1992) in a optimal stopping problem, also focusing on the "importance of the real option to retire for portfolio choice".

In solving our problem we must assume that the market is complete. This means that the agent is able to borrow against his future wages (human capital) and any risk on the financial market can be perfectly hedged through a suitable portfolio.

The value function of the second problem, that is the optimal stopping problem, can be written in closed form as a function of a wage threshold. When this threshold is crossed, then the agent finds it optimal to retire.

Finally, we calibrate our model to the US data and we present some numerical results that allow to investigate the dynamic behaviour of the optimal consumption, labour supply, and portfolio.

We show that the optimal retirement age should be between 50 and 65. Our optimal retirement age is always below the statutory value with a minimum deviation between 0 and 3 years.

Furthermore, we show that the optimal (relative) consumption increases over time from a ratio of about 2.5%. Then, once the retirement is reached, it becomes a constant percentage (5.5%) of the disposable wealth. Moreover, we find that the percentage of disposable wealth invested in the risky asset is increasing over time, but quite stable in a range between 57% and 58%, while the portfolio volatility is more volatile at the beginning of the working life and reaches a minimum at the retirement age.

Finally, the optimal average labour supply is decreasing over time. Between zero and about 25 years after starting working, the agent supplies a stable amount of hours per year. Then, this number of hours start decreasing and, on average, it reaches zero after 50 years from the beginning of agent's working life.

The remaining of the paper is organised as follows: Section 2 show the framework, Section 3 presents the result of the first problem, i.e. the optimal consumption, labour supply, and portfolio. Section 4 shows the optimal stop-

[2] For a summary of the literature see Dixit and Pindyck, 1994.

ping problem, and solves it in closed form. Section 5 presents a numerical simulation calibrated on the US data. Some technical results are gathered in two appendices.

2 The model setup

2.1 Financial Market

On a continuously open and friction-less financial market over the time set $[t_0, +\infty[$, one risky assets is traded, and its price $S_t \in \mathbb{R}_+$ follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{1}$$

where both the expected return (μ) and the volatility (σ) are constant, and W_t is a Wiener processes (with zero mean and variance t). The initial asset price S_{t_0} is known. Also a risk-less asset is listed and its price G_t solves the ordinary differential equation

$$\frac{dG_t}{G_t} = rdt, \tag{2}$$

where r is the risk-less interest rate. We assume $G_{t_0} = 1$, i.e. the risk-less asset is the *numéraire* of the economy.

This financial market is arbitrage free and complete. In other words, there exists a unique market price of risk ξ such that $\sigma\xi = \mu - r$.

Girsanov's theorem allows us to switch from the historical ($\mathbb{P}$) to the risk-neutral probability $\mathbb{Q}$ by using $dW_t^{\mathbb{Q}} = \xi dt + dW_t$. The value in t_0 of any cash flow Ξ_t available at time t can be written as

$$\Xi_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\Xi_t \frac{G_{t_0}}{G_t} \right] = \mathbb{E}_{t_0}^{\mathbb{Q}} \left[\Xi_t e^{-r(t-t_0)} \right] = \mathbb{E}_{t_0} \left[\Xi_t m_{t_0,t} e^{-r(t-t_0)} \right], \tag{3}$$

where $\mathbb{E}_{t_0}[\bullet]$ and $\mathbb{E}_{t_0}^{\mathbb{Q}}[\bullet]$ are the expected value operators under the historical ($\mathbb{P}$) and the risk neutral ($\mathbb{Q}$) probabilities respectively, conditional on the information set at time t_0, and the martingale $m_{t_0,t}$, such that $m_{t_0,t_0} = 1$, solves

$$\frac{dm_{t_0,t}}{m_{t_0,t}} = -\xi dW_t. \tag{4}$$

2.2 Agent's wage

In this framework, we assume that the agent (household) works for the firm whose stocks are listed on the financial market, and this firm pays an instantaneous wage w_t to the agent. Thus, the risk source that drives the price S_t is the same that drives the wage. Accordingly, we assume that w_t solves the following differential equation:

$$\frac{dw_t}{w_t} = \mu_w dt + \sigma_w dW_t. \tag{5}$$

The agent can choose how much to work (l_t) at any instant and, accordingly, his instantaneous wage will be $l_t w_t$. The agent optimally chooses when to retire (at time T), and after retirement he will not receive the wage w_t any longer.

2.3 The mortality risk

The agent is aged x at t_0 and he dies at a random time $\omega \in [t_0, \infty[$. If we call λ_t the force of mortality, the probability to be alive at time t, given that he is alive at time t_0, is

$$\mathbb{P}\{\omega > t | \omega > t_0\} = \mathbb{E}_{t_0}[\mathbb{I}_{\omega>t}] = e^{-\int_{t_0}^{t} \lambda_u du}, \tag{6}$$

where $\mathbb{I}_\varepsilon$ is the indicator function of the event ε whose value is 1 if the event happens and 0 otherwise.

2.4 The human capital

In this framework, the human capital is defined as the expected present value of future wages that the agent will obtain during his working life. If we define Ψ_{t_0} such a human capital at time t_0, we can compute it as follows:

$$\begin{aligned}\Psi_{t_0}(T) &:= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty} w_t \mathbb{I}_{t<T} e^{-\int_{t_0}^{t} \lambda_s ds - r(s-t_0)} dt\right] \\ &= \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} w_t m_{t_0,t} \mathbb{I}_{t<T} e^{-\int_{t_0}^{t} \lambda_s ds - r(s-t_0)} dt\right].\end{aligned} \tag{7}$$

The human capital is computed under the risk neutral probability since it measures a kind of "market power" for the agent. In the complete market of our framework, the agent can trade his human capital. Accordingly, his disposable income is not only the current wage, but also the future flow of wages.

If we use the property of the indicator function $\mathbb{I}_{t<T} = 1 - \mathbb{I}_{t\geq T}$, the human capital can be written as

$$\begin{aligned}\Psi_{t_0}(T) =& \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} w_t m_{t_0,t} e^{-\int_{t_0}^{t} \lambda_s ds - r(s-t_0)} dt\right] \\ &- \mathbb{E}_{t_0}\left[\int_{T}^{\infty} w_t m_{t_0,t} e^{-\int_{t_0}^{t} \lambda_s ds - r(s-t_0)} dt\right],\end{aligned}$$

in which the subtrahend is the human capital which the agent renounces to when he decides to retire.

2.5 The investor's wealth

The investor holds a portfolio given by $\theta_{G,t} \in \mathbb{R}$ quantities of the risk-less asset and $\theta_t \in \mathbb{R}$ quantities of the risky asset. Thus, at any instant in time the value of his wealth R_t is given by

$$R_t = \theta_{G,t} G_t + \theta_t S_t. \tag{8}$$

Negative values of $\theta_{G,t}$ and θ_t indicate short selling. The differential of this constraint is

$$dR_t = \theta_{G,t} dG_t + \theta_t dS_t + \left(d\theta_{G,t}\left(G_t + dG_t\right) + d\theta_t\left(S_t + dS_t\right)\right), \tag{9}$$

where the term in brackets must:

- finance the agent's consumption $c_t dt$,
- be financed by the agent's salary during his working life $w_t l_t \mathbb{I}_{t<T} dt$,
- finance the loss in wealth due to death: $\lambda_t R_t dt$.

Thus, the dynamic constraint can be written as

$$dR_t = \theta_{G,t} dG_t + \theta_t dS_t - c_t dt + w_t l_t \mathbb{I}_{t<T} dt + \lambda_t R_t dt,$$

and if $\theta_{G,t}$, dS_t, and dG_t are substituted from (8), (1), and (2) respectively, we obtain

$$dR_t = \left(R_t\left(r + \lambda_t\right) + \theta_t S_t\left(\mu - r\right) - c_t + w_t l_t \mathbb{I}_{t<T}\right) dt + \theta_t S_t \sigma dW_t. \tag{10}$$

Note that under the risk neutral probability, (10) can be written as

$$\begin{aligned} dR_t &= \left(R_t\left(r + \lambda_t\right) + \theta_t S_t\left(\mu - r\right) - c_t + w_t l_t \mathbb{I}_{t<T}\right) dt + \theta_t S_t \sigma \left(dW_t^{\mathbb{Q}} - \xi dt\right) \\ &= \left(R_t\left(r + \lambda_t\right) - c_t + w_t l_t \mathbb{I}_{t<T}\right) dt + \theta_t S_t \sigma dW_t^{\mathbb{Q}}, \end{aligned}$$

and it implies that the initial wealth must be

$$R_{t_0} = \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty}\left(c_t - w_t l_t \mathbb{I}_{t<T}\right) e^{-\int_{t_0}^{t} r + \lambda_u du} dt\right],$$

and this is actually the constraint that we will use for solving the optimisation problem of the agent.

2.6 Utility function

The utility of the agent is assumed to be additive in three components.

1. The inter-temporal utility of consumption (c_t) that is obtained by the agent over his whole lifetime. If the agent's preferences belong to the Hyperbolic Absolute Risk Aversion (HARA) family, then this utility component can be written as
$$\frac{\left(c_t - c_m\right)^{1-\delta}}{1-\delta},$$
in which the parameter δ measures the (relative) risk aversion, and c_m can be interpreted as a minimum (subsistence) level of consumption.

2. The inter-temporal utility of leisure, which is added to the previous utility during the working lifetime. In particular, we assume that this component can be written, at any instant in time, as

$$\chi_A \frac{(L - l_t)^{1-\delta}}{1 - \delta} \mathbb{I}_{t<T},$$

in which δ is the same risk aversion parameters as before, L is the maximum number of working hours that the agent can supply and, thus, the difference $L - l_t$ is the leisure time. The constant parameter χ_A measures the relative importance of the utility of leisure with respect to the utility of consumption.

3. The utility obtained at retirement, that we assume to be proportional to the remaining human capital of the agent. This utility component is given by

$$\chi_B \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} w_t m_{t,s} \mathbb{I}_{t \geq T} e^{-\int_{t_0}^{t} \lambda_u du - r(t-t_0)} dt \right],$$

in which χ_B measures the relative importance of the gain in utility due to retirement. Here, the hypothesis is that the retirement is enjoyed as much as the human capital that "remains". If the agent chooses to retire when he is still young (old), and so his remaining human capital is still high (low), then he enjoys more (less) his leisure. Of course, the counterpart of this choice is that he has to renounce to the future wages and thus the wealth can accumulate at a lower speed and the consumption will have to be reduced accordingly.

Finally, the optimisation problem can be written as follows:

$$\max_{\{c_t, l_t, \theta_t\}_{t \in [t_0, \infty[}, T} \mathbb{E}_{t_0} \left[\int_{t_0}^{\infty} \left(\frac{(c_t - c_m)^{1-\delta}}{1 - \delta} + \chi_A \frac{(L - l_t)^{1-\delta}}{1 - \delta} \mathbb{I}_{t<T} \right) e^{-\rho(t-t_0) - \int_{t_0}^{t} \lambda_s ds} dt \right] \tag{11}$$
$$+ \mathbb{E}_{t_0} \left[\chi_B \int_{t_0}^{\infty} w_t m_{t_0,t} \mathbb{I}_{t \geq T} e^{-\int_{t_0}^{t} \lambda_u du - r(t-t_0)} dt \right],$$

in which ρ is a subjective discount factor.

3 The optimal consumption, portfolio, and labour

In order to compute the optimal solution to Problem (11), we split it into two sub-problems. The idea is to follow a three step procedure.

1. Optimising with respect to consumption, labour supply, and portfolio, given any possible pension time T. In optimising this part of the problem,

we treat T as a stochastic variable. The last component in Problem (11) does not depend on c_t, l_t, or θ_t, and, accordingly, it is neglected in solving this first step.

2. Computing the value function of the problem as a function of the stochastic variable T.

3. Optimising the value function with respect to T.

The solution of the first step is shown in the following proposition.

Proposition 1. *Given the stochastic pension time T, the optimal consumption, portfolio, and labour supply that solve Problem (11) given (5) and (10) are*

$$c_t^* = c_m + \frac{R_t - H_t}{F_t}, \tag{12}$$

$$l_t^* = L - \left(\frac{\chi_A}{w_t}\right)^{\frac{1}{\delta}} \frac{R_t - H_t}{F_t}, \tag{13}$$

$$S_t\theta_t^* = \frac{R_t - H_t}{\delta}\frac{\xi}{\sigma} + \frac{\sigma_w}{\sigma}\left(\frac{\partial H_t}{\partial w_t} + \frac{R_t - H_t}{F_t}\frac{\partial F_t}{\partial w_t}\right), \tag{14}$$

where

$$H_t := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\infty \left(c_m - L w_s \mathbb{I}_{s<T}\right) e^{-r(s-t)-\int_t^s \lambda_u du} ds\right], \tag{15}$$

$$F_t := \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^\infty \left(\chi_A^{\frac{1}{\delta}} w_s^{1-\frac{1}{\delta}} \mathbb{I}_{s<T} + 1\right) e^{-\left(\frac{\delta-1}{\delta}r + \frac{1}{\delta}\rho + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t) - \int_t^s \lambda_u du} ds\right], \tag{16}$$

and

$$dW_t^{\mathbb{Q}_\delta} = \frac{\delta - 1}{\delta}\xi dt + dW_t.$$

Proof. See Appendix A. □

The optimal consumption (12) implies that it is always optimal to consume more than the subsistence level. In fact, we have demonstrated in Appendix A that the ratio $\frac{R_t - H_t}{F_t}$ is always positive.

Here, the function H_t measures the expected present value of all the future subsistence consumption (c_m) reduced by the expected present value of the future maximum, or potential, wage (i.e. the wage that the agent would receive if he supplied the maximum number of labour L). This result implies that if c_m is very high, then the optimal consumption will be close to this minimum level. In fact, if the consumption is sufficiently low, the wealth can be accumulated at a higher rate.

The optimal labour supply (13) is lower than the maximum level L and the reduction is inversely proportional to the wage and directly proportional to the ratio $\frac{R_t - H_t}{F_t}$. The elasticity of the labour supply with respect to w_t is not trivial, since both function H_t and F_t depend on w_t.

Nevertheless, by combining (12) and (13) we can write

$$\left(l_t^* - L\right) = -\left(\frac{\chi_A}{w_t}\right)^{\frac{1}{\delta}} \left(c_t^* - c_m\right),$$

from which we obtain

$$\frac{\partial\left(l_t^* - L\right)}{\partial w_t}\frac{1}{\left(l_t^* - L\right)} = \frac{\partial\left(c_t^* - c_m\right)}{\partial w_t}\frac{1}{\left(c_t^* - c_m\right)} - \frac{1}{\delta}\frac{1}{w_t},$$

where we can conclude what follows.

Corollary 2. *The semi-elasticity of the optimal labour supply with respect to wage is always lower than the same semi-elasticity of consumption.*

The optimal portfolio (14) is formed by three components: (i) a speculative component $\frac{R_t - H_t}{\delta}\frac{\xi}{\sigma}$ which is proportional to the market price of risk ξ and negatively depends on both the risk aversion δ and the volatility σ, (ii) a hedging component that covers the agent against the stochastic changes in H_t due to the change in the wage w_t, and (iii) a hedging component that covers against the stochastic changes in F_t due to the change in the wage w_t.

We note that the function H_t can be simplified as follows

$$\begin{aligned} H_t &= c_m \int_t^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds - L\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^\infty w_s \mathbb{I}_{s<T} e^{-r(s-t)-\int_t^s \lambda_u du} ds\right] \\ &= c_m \int_t^\infty e^{-r(s-t)-\int_t^s \lambda_u du} ds - L\Psi_t\left(T\right) \end{aligned}$$

where only the second term depends on w_t, and actually coincides with the human capital defined in (7). Also the function F_t can be simplified in a similar way:

$$\begin{aligned} F_t =& \chi_A^{\frac{1}{\delta}} \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^\infty w_s^{1-\frac{1}{\delta}} \mathbb{I}_{s<T} e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t)-\int_t^s \lambda_u du} ds\right] \\ &+ \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^\infty e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t)-\int_t^s \lambda_u du} ds\right], \end{aligned}$$

and also in this case F_t depends on w_t only through the first term.

Accordingly, the optimal portfolio can be alternatively written as

$$\begin{aligned} S_t\theta_t^* =& \frac{R_t - H_t}{\delta}\frac{\xi}{\sigma} - \frac{\sigma_w}{\sigma} L \frac{\partial \Psi_t\left(T\right)}{\partial w_t} \\ &+ \frac{\sigma_w}{\sigma}\frac{R_t - H_t}{F_t}\chi_A^{\frac{1}{\delta}} \frac{\partial \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^T w_s^{1-\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t)-\int_t^s \lambda_u du} ds\right]}{\partial w_t}. \end{aligned}$$

3.1 The value function

The value function of Problem (11) is obtained by substituting the optimal consumption and labour supply $(c_t^*,\, l_t^*)$ into the objective function:

$$\begin{aligned} & J\left(T|\, t_0, w_{t_0}, R_{t_0}\right) \\ =& \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} \frac{1}{1-\delta}\left(\frac{R_t-H_t}{F_t}\right)^{1-\delta}\left(1+\frac{\chi_A}{1-\delta} w_t^{1-\frac{1}{\delta}} \mathbb{I}_{t<T}\right) e^{-\rho(t-t_0)-\int_{t_0}^t \lambda_s ds} dt\right] \\ & + \chi_B \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} w_t m_{t_0,t} \mathbb{I}_{t \geq T} e^{-\int_{t_0}^t \lambda_u du - r(t-t_0)} dt\right]. \end{aligned} \tag{17}$$

The value of the ratio $\left(R_t - H_t\right)/F_t$ can be written as a function of the initial wealth

$$\frac{R_t-H_t}{F_t}=\frac{R_{t_0}-H_{t_0}}{F_{t_0}} m_{t_0,t}^{-\frac{1}{\delta}}\left(\frac{e^{-r(t-t_0)}}{e^{-\rho(t-t_0)}}\right)^{-\frac{1}{\delta}},$$

and, accordingly, the value function becomes

$$J\left(T|\, t_0, w_{t_0}, R_{t_0}\right)=F_{t_0}^{\delta} \frac{\left(R_{t_0}-H_{t_0}\right)^{1-\delta}}{1-\delta}+\chi_B \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} w_t m_{t_0,t} \mathbb{I}_{t \geq T} e^{-\int_{t_0}^t \lambda_u du - r(t-t_0)} dt\right]$$

in which H_{t_0} and F_{t_0} do contain the pension time T (see (15) and (16)).

4 The optimal pension time problem

The optimal pension time can be found by solving the problem

$$\max_T J\left(T|\, t_0, w_{t_0}, R_{t_0}\right),$$

where the value function J has been defined in (17).

The computation of the optimal stopping time can be crucially simplified is we further assume that the force of mortality (λ) is constant. In this case, in fact, we are able to find a closed form solution to the value function J.

The two stochastic processes that we need to compute the expected values in the functions H_t and F_t are $m_{t_0,t}w_t$ and $\left(m_{t_0,t}w_t\right)^{1-\frac{1}{\delta}}$. Given the stochastic processes (4) and (5), we can immediately write:

$$\frac{dw_t^{1-\frac{1}{\delta}}}{w_t^{1-\frac{1}{\delta}}}=\left(1-\frac{1}{\delta}\right)\left(\mu_w-\frac{1}{2}\frac{1}{\delta}\sigma_w^2\right)dt+\left(1-\frac{1}{\delta}\right)\sigma_w dW_t,$$

while $m_{t_0,t}^{1-\frac{1}{\delta}}$ solves

$$\frac{dm_{t_0,t}^{1-\frac{1}{\delta}}}{m_{t_0,t}^{1-\frac{1}{\delta}}}=-\frac{1}{2}\left(1-\frac{1}{\delta}\right)\frac{1}{\delta}\xi^2 dt-\left(1-\frac{1}{\delta}\right)\xi dW_t,$$

from which

$$\frac{d\left(m_{t_0,t}w_t\right)^{1-\frac{1}{\delta}}}{\left(m_{t_0,t}w_t\right)^{1-\frac{1}{\delta}}} = \left(1-\frac{1}{\delta}\right)\left(\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \left(1-\frac{1}{\delta}\right)\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2\right)dt + \left(1-\frac{1}{\delta}\right)\left(\sigma_w - \xi\right)dW_t,$$

and, of course, by taking the limit for $\delta \to \infty$:

$$\frac{d\left(m_{t_0,t}w_t\right)}{\left(m_{t_0,t}w_t\right)} = \left(\mu_w - \xi\sigma_w\right)dt + \left(\sigma_w - \xi\right)dW_t.$$

Accordingly, the functions H_{t_0} and F_{t_0} can be simplified as follows:

$$\begin{aligned} H_{t_0} &= \mathbb{E}_{t_0}^{\mathbb{Q}}\left[\int_{t_0}^{\infty}\left(c_m - Lw_s\mathbb{I}_{s<T}\right)e^{-(r+\lambda)(s-t_0)}ds\right] \\ &= \frac{c_m}{r+\lambda} - L\mathbb{E}_{t_0}\left[\int_{t_0}^{T} m_{t_0,s}w_s e^{-(r+\lambda)(s-t_0)}ds\right], \end{aligned}$$

and

$$\begin{aligned} F_{t_0} &= \mathbb{E}_{t_0}^{\mathbb{Q}_\delta}\left[\int_{t_0}^{\infty}\left(\chi_A^{\frac{1}{\delta}}w_s^{1-\frac{1}{\delta}}\mathbb{I}_{s<T}+1\right)e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\lambda+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t_0)}ds\right] \\ &= \frac{1}{\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\lambda+\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2} + \chi_A^{\frac{1}{\delta}}\mathbb{E}_{t_0}\left[\int_{t_0}^{T}\left(m_{t_0,s}w_s\right)^{1-\frac{1}{\delta}}e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho+\lambda\right)(s-t_0)}ds\right]. \end{aligned}$$

Finally, the last part of the value function can be simplified as follows:

$$\chi_B\mathbb{E}_{t_0}\left[\int_{T}^{\infty} w_t m_{t_0,t}e^{-(r+\lambda)(t-t_0)}dt\right].$$

Since now all the implied stochastic processes are geometric Brownian motions, we can use standard techniques for rewriting the optimal stopping problem into an optimal threshold problem. To this aim, we rely on the following result.

Proposition 3. *Given the stochastic process*

$$\frac{dX_t}{X_t} = \alpha dt + \beta dW_t,$$

and a stopping time T, with $X_T = x$, the following equations hold if $\rho > \alpha$:

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{T} X_s e^{-\rho(s-t_0)}ds\right] = \frac{X_{t_0} - x^{1-\gamma}X_{t_0}^{\gamma}}{\rho-\alpha},$$

$$\mathbb{E}_{t_0}\left[\int_{T}^{\infty} X_s e^{-\rho(s-t_0)}ds\right] = \frac{X_{t_0}}{\rho-\alpha} - \mathbb{E}_{t_0}\left[\int_{t_0}^{T} X_s e^{-\rho(s-t_0)}ds\right] = \frac{x^{1-\gamma}X_{t_0}^{\gamma}}{\rho-\alpha},$$

in which

$$\gamma = -\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)+\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2\rho}{\sigma^2}}.$$

Proof. See Appendix B. □

As it is common in the literature about optimal stopping time, we have now transformed the original optimal stopping problem into an optimal threshold problem where we want to find a value of the state variable (the wage) such that the value function is maximum. The optimal stopping time, then, is given by the first moment when the wage crosses the threshold.

Given the above results, the value function can be further simplified as follows, where κ is the threshold of the wage (i.e. $\kappa := w_T$):

$$J(\kappa) = F_{t_0}^{\delta} \frac{(R_{t_0} - H_{t_0})^{1-\delta}}{1-\delta} + \chi_B \frac{\kappa^{1-\gamma_1} w_{t_0}^{\gamma_1}}{\rho + \lambda - \mu_w + \xi\sigma_w},$$

in which

$$H_{t_0} = \frac{c_m}{r+\lambda} - L \frac{w_{t_0} - \kappa^{1-\gamma_1} w_{t_0}^{\gamma_1}}{r + \lambda - \mu_w + \xi\sigma_w},$$

$$\begin{aligned} F_{t_0} =& \frac{1}{\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \lambda + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2} \\ &+ \chi_A^{\frac{1}{\delta}} \frac{w_{t_0}^{1-\frac{1}{\delta}} - \left(\kappa^{1-\frac{1}{\delta}}\right)^{1-\gamma_2} \left(w_{t_0}^{1-\frac{1}{\delta}}\right)^{\gamma_2}}{\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \lambda - \frac{\delta-1}{\delta}\left(\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \left(1-\frac{1}{\delta}\right)\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2\right)}, \end{aligned}$$

and

$$\gamma_1 := \frac{1}{2} - \frac{\mu_w - \xi\sigma_w}{(\sigma_w - \xi)^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu_w - \xi\sigma_w}{(\sigma_w - \xi)^2}\right)^2 + 2\frac{r+\lambda}{(\sigma_w - \xi)^2}},$$

$$\begin{aligned} \gamma_2 :=& \frac{1}{2} - \frac{\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \frac{\delta-1}{\delta}\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2}{\frac{\delta-1}{\delta}(\sigma_w - \xi)^2} \\ &+ \sqrt{\left(\frac{1}{2} - \frac{\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \frac{\delta-1}{\delta}\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2}{\frac{\delta-1}{\delta}(\sigma_w - \xi)^2}\right)^2 + 2\frac{\frac{\delta-1}{\delta}r + \frac{1}{\delta}\rho + \lambda}{\left(\frac{\delta-1}{\delta}\right)^2(\sigma_w - \xi)^2}}. \end{aligned}$$

The expected values in the functions F_t and H_t converge only if

$$r + \lambda - \mu_w + \xi\sigma_w > 0$$

and

$$\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \lambda - \frac{\delta-1}{\delta}\left(\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \left(1-\frac{1}{\delta}\right)\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2\right) > 0.$$

5 The calibration of the model

5.1 The parameters

The parameters of the risky asset (μ and σ) are calibrated on the daily values of S&P500 from 1970 to 2018. We apply the method of moments to the geometric Brownian motion and we obtain

$$\mu = 0.084, \quad \sigma = 0.167.$$

The wage process is calibrated on the wages and salaries for US workers (fred.stlouisfed.org series A576RC1A027NBEA) for the same period (1970–2018). The same method of moment gives

$$\mu_w = 0.0612, \quad \sigma_w = 0.07.$$

The initial value of the wage is set to 20 dollars (per hour). Of course this value can be changed without any loss of generality, cause it affects just the level of the value function but not its shape.

The risk-less return is assumed to coincide with the average return on 3 month T-Bill from 1950 to 2018:

$$r = 0.0437.$$

For the sake of simplicity we further assume that the subjective discount rate ρ coincides with this risk-less interest rate.

The force of mortality is calibrated on the data of the human mortality database (www.mortality.org) for US, on the average of both males and females aged 25:

$$\lambda = 0.0038.$$

Here, we have used all annual data (interest rates and growth rates), and so the maximum number of working hours is set to $L = 250 \times 8$, given by the product between the number of working days in a year and the number of working hours in a day.

The initial wealth is set to a level that coincides with the total amount that could be received by working the maximum of hours in a year at the initial wage (i.e. Lw_{t_0}).

The value of χ_B is crucial to the problem. In fact, its value strongly affects the shape of the value function $J(\kappa)$. In particular:

- if $\chi_B = 0$, i.e. there is no incentive in retiring, then the value function is constantly increasing over κ and, in fact, it is never optimal to retire;
- if χ_B is sufficiently high, then the value function is constantly decreasing and it is optimal to retire as soon as possible, since the utility from retirement is over-weighted with respect to the utility of leisure; of course, this threshold of χ_B depends on the values of all the other parameters of the model;

Table 1: Parameters of the model

Financial market	Wage	Subjective param.	Survival
μ =0.084	μ_w =0.0612	$\rho = r$ =0.0437	λ =0.0038
σ =0.167	σ_w =0.07	δ =2.5	
r =0.0437	w_{t_0} =20	c_m =0	
$\xi = \frac{\mu - r}{\sigma}$ =0.2413174	L =2000	$R_{t_0} = w_{t_0} L$ =4×10^4	
		$\chi_A = w_{t_0}^{1-\delta}$ =0.0111803	
		$\chi_B = 5 \times 10^{-9}$	

- if χ_B is between 0 and the previous threshold, then the value function is concave, and there exists a finite optimal stopping time. The value of χ_B that we chose for our framework is 5×10^{-9}, since it is consistent with the other parameters and it allows to make a comparison between the units of measure of both the total expected present value of all the future wages and the inter-temporal utility function of consumption and leisure.

The values of all the parameters are gathered in Table 1.

5.2 The value function and the optimal wage threshold

The numerical values of the value function $J(\kappa)$, in which κ is the wage threshold, are drawn in Figure 1. The maximum of the function is obtained for

$$\kappa = 119.382,$$

which is the threshold of the wage above which the agent will decide to retire. If we simulate some trajectories of the wage and we compare it with this threshold, we obtain the graph shown in Figure 2.

The time zero in the graph represents the first period when the agent starts receiving a wage. Thus, it should coincide with an age of about 25.

The main result is that, in our framework, it is optimal for an agent to retire on average after about 30 years of work. Nevertheless, the volatility of the wage, implies that this decision may be either anticipated to about 25 years of work or postponed up to more that 40 years. Of course if the wage increases at a higher (lower) rate, then it is optimal to retire earlier (later). Therefore, according to our framework, the optimal retirement age should be between 50 and 65, with an average value of around 55 years.

If we compare this result with the EU15 average retirement age of 65 years and with the trend forecast for the next few years (towards 68 years for some states), we can identify a misalignment between the statutory and the optimal retirement age.

Figure 1: Value function $J(\kappa)$ given the parameters shown in Table 1

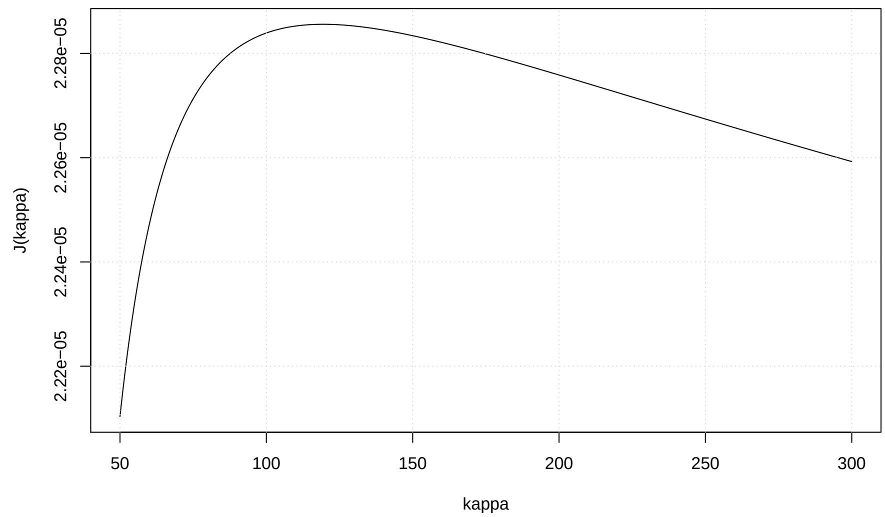
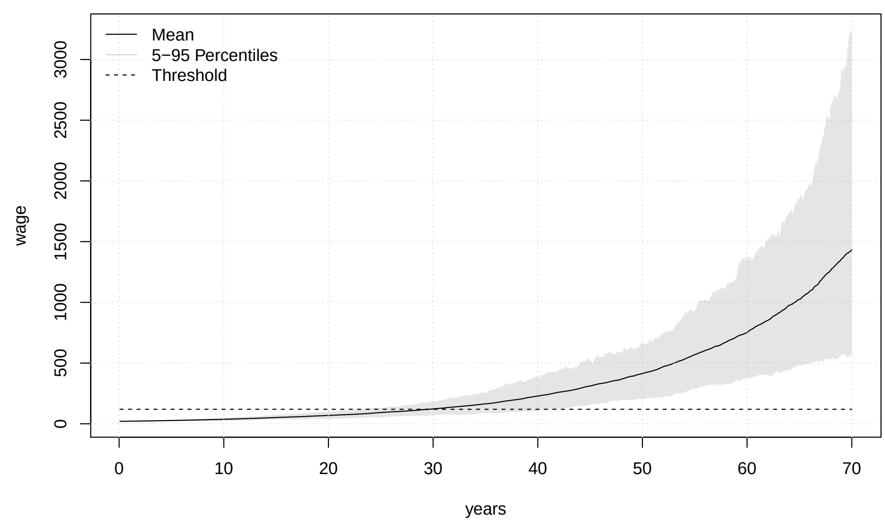


Figure 2: Optimal stopping time as the moment when the wage (simulated in light-grey with 100 trajectories) goes above the (dotted) threshold κ. In light-grey the area between the 5 and 95 percentiles of the simulations



In our framework, we do not take into account any pension scheme and, thus, we do not have to comply with the related sustainability issue. Instead, in our both public and private pension systems in Europe, such an issue is becoming more and more relevant over time, especially in the face of an ageing population and a reduced birth rate trend

Another hypothesis relates instead to the growth rate of wages. Indeed, if the wages increase at a lower rate, as shown, among others, by IMF (2018), European Commission (2018a), and Astrov et al. (2018), then it is optimal to retire later. A trend reduction in the average growth rate o wages could lead to a slow increase and alignment between the statutory and optimal retirement age.

5.3 The auxiliary functions

Under the hypotheses of our framework, the function H_t is always negative (or zero) and has the following value

$$H_t = -L \frac{w_t - \kappa^{1-\gamma_1} w_t^{\gamma_1}}{r + \lambda - \mu_w + \xi \sigma_w} \mathbb{I}_{w_t < \kappa}. \tag{18}$$

We recall that this function measures the (opposite of the) expected present value of the future wages. The value, over time, of this function is drawn in Figure 3.

The function F_t is always strictly positive and has the following value

$$\begin{aligned} F_t = & \frac{1}{\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \lambda + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2} \\ & + \chi_A^{\frac{1}{\delta}} \frac{w_t^{1-\frac{1}{\delta}} - \left(\kappa^{1-\frac{1}{\delta}}\right)^{1-\gamma_2} \left(w_t^{1-\frac{1}{\delta}}\right)^{\gamma_2}}{\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \lambda - \frac{\delta-1}{\delta}\left(\mu_w - \frac{1}{2}\frac{1}{\delta}\sigma_w^2 - \left(1-\frac{1}{\delta}\right)\xi\sigma_w - \frac{1}{2}\frac{1}{\delta}\xi^2\right)} \mathbb{I}_{w_t < \kappa}. \end{aligned} \tag{19}$$

The value, over time, of this function is drawn in Figure 4.

5.4 The optimal wealth, consumption, labour, and portfolio

In Figure 5 we draw the optimal wealth. We check that the optimal wealth is increasing over time, but before retirement the growth rate is a bit higher than the growth rate after retirement. This obvious result is due to the fact that, after retirement, the agent does not receive a wage any longer.

In Figure 6 we draw the optimal labour supply. We see that the average labour supply is stable over a period of about 25 working years. After this period, for some simulated paths the agent starts retiring and so the average labour supply reduces over time. The stable value of 500 working hours per year, which may seem low, depends on two crucial hypotheses on financial market and agent's preferences: (i) the market does not suffer any credit risk, in other words, the asset prices have some volatility, but they never go to zero, and the

Figure 3: Result of 100 simulations of function H_t over time in (18). The grey curves are the 5 and the 95 percentiles

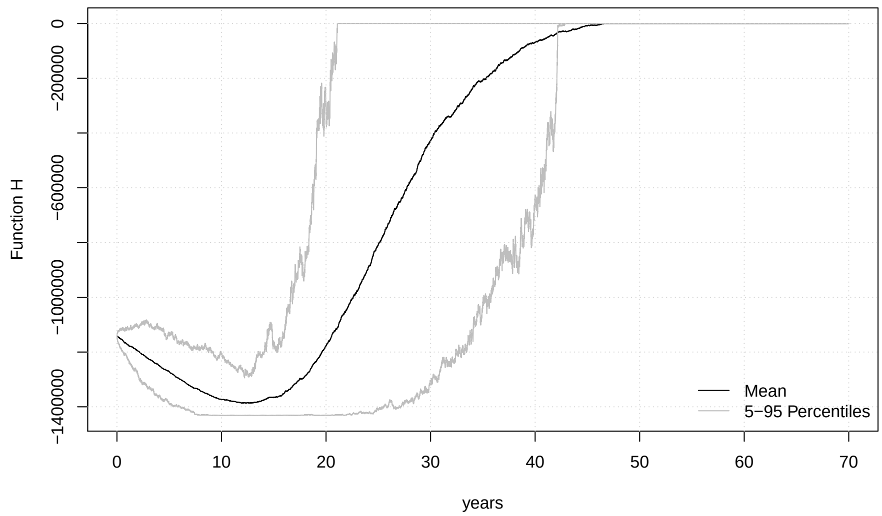


Figure 4: Result of 100 simulations of function F_t over time in (19). The grey curves are the 5 and the 95 percentiles

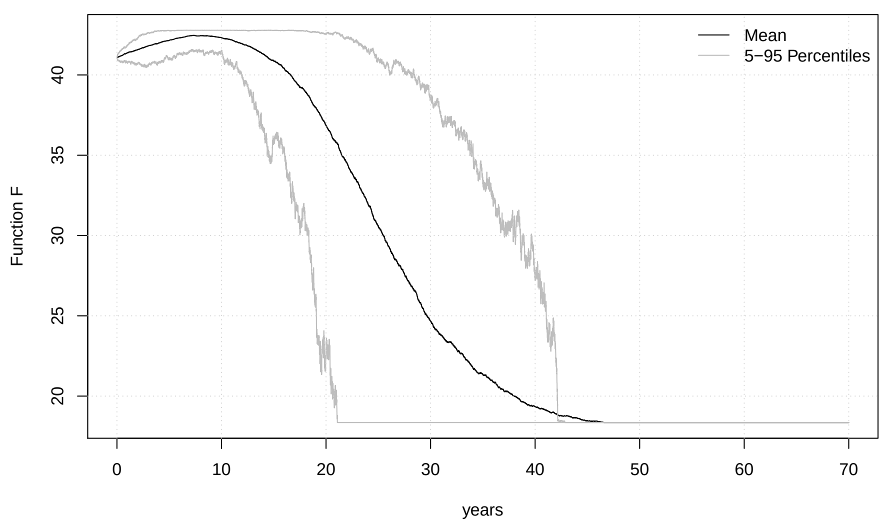


Figure 5: Result of 100 simulations of the optimal wealth. In light-grey the area between the 5 and 95 percentiles of the simulations

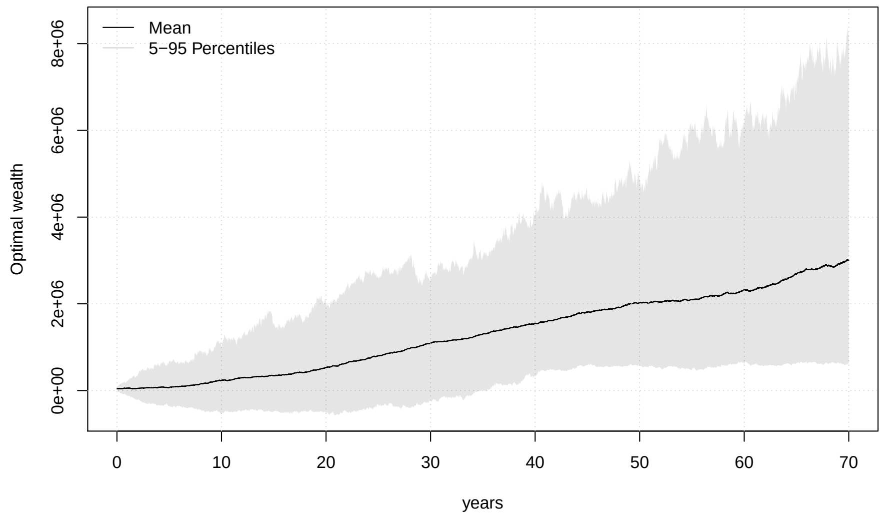


Figure 6: Result of 100 simulations of the optimal labour supply l_t^*. The grey curves are the 5 and the 95 percentiles

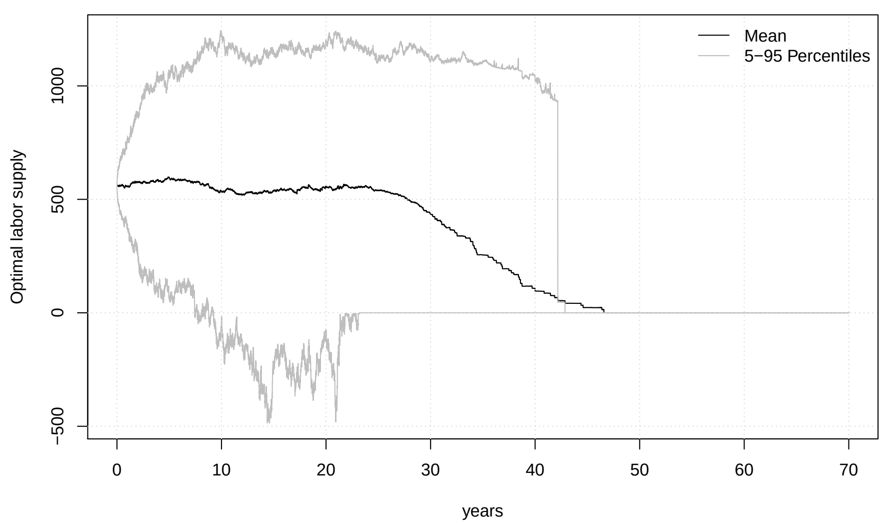
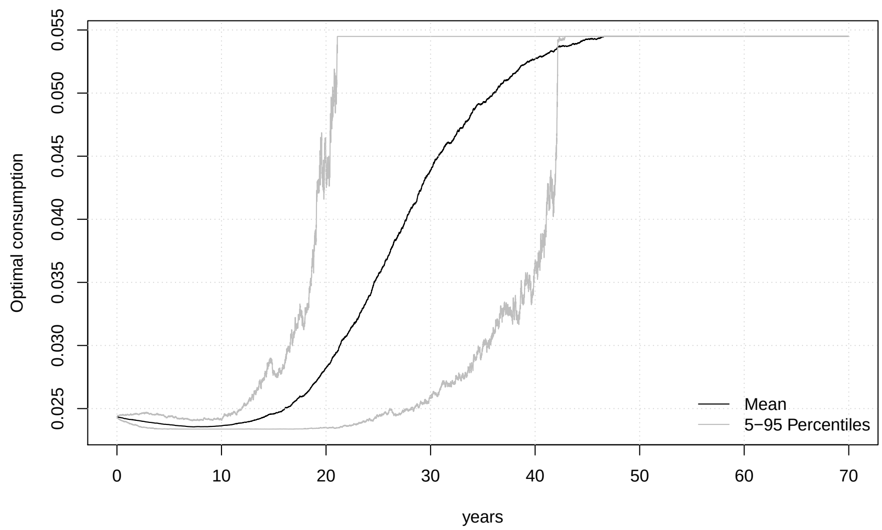


Figure 7: Result of 100 simulations of the optimal consumption as a percentage of disposable wealth $\frac{c_t^*}{R_t - H_t}$. The grey curves are the 5 and the 95 percentiles



financial market is always able to fully recover from any fall, (ii) the agent has a subsistence consumption $c_m = 0$, which means that he is able to take utility even from a very low consumption.

The optimal consumption of the agent can be expressed in different ways: (i) as a total amount of money spent, (ii) as a percentage of the optimal wealth R_t, or (iii) as a percentage of the disposable wealth $R_t - H_t$. This last way to represent consumption seems to be the best since it suitably takes into account the implicit hypothesis that the agent is able to borrow against his future wage. Figure 7 shows the result of 100 simulations of the ratio $\frac{c_t^*}{R_t - H_t}$.

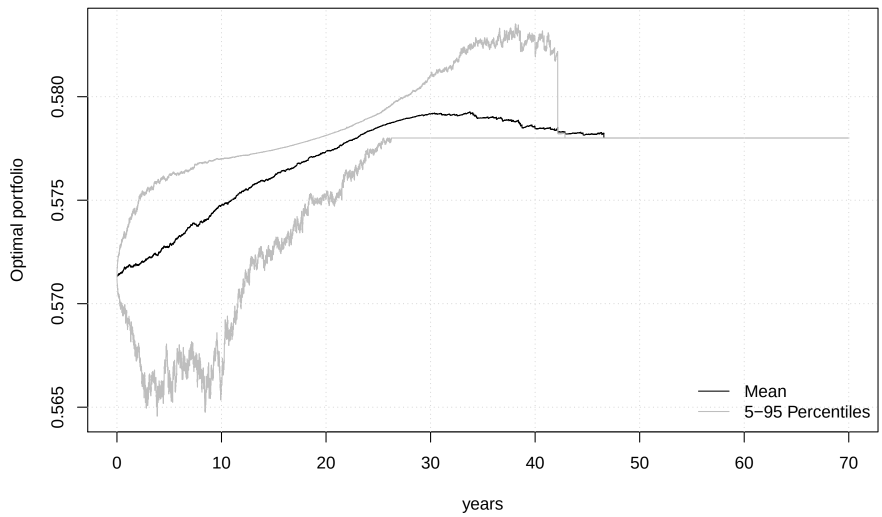
We see that the optimal (relative) consumption increases over time and once the retirement is reached, it becomes a constant percentage of the disposable wealth. The optimal consumption ratio starts from a value of about 2.5% and reaches, after retirement, its asymptotic value of about 5.5%.

The optimal portfolio can be drawn, again, as a percentage of the disposable wealth $\frac{\theta_t^*}{R_t - H_t}$. The corresponding plot is shown in Figure 8.

We can see some interesting behaviours of the optimal portfolio share:

- the percentage of disposable wealth invested in the risky asset is increasing over time, but quite stable in a range between 57% and 58%
- the optimal portfolio is more volatile at the beginning of the working life and this is due to the need to hedge against the volatility of future wage

Figure 8: Result of 100 simulations of the optimal portfolio as a percentage of disposable wealth $\frac{\theta_t^*}{R_t - H_t}$. The grey curves are the 5 and the 95 percentiles



- the volatility of the portfolio is at its minimum close to the retirement age
- this volatility starts increasing again until the last agent (of the simulated paths) retires
- after retirement there is no more need for hedging against wage volatility, and the optimal portfolio share can become a constant percentage of disposable wealth.

6 Conclusion

In our work we study the problem of a representative agent who wants to maximise the expected present value of his inter-temporal utility by choosing the retirement age, the inter-temporal consumption, the labour supply, and the portfolio allocation.

If we assume that an agent starts working at about 25, our main results show that the optimal retirement age should be between 50 and 65, with an average value of around 55 years. Comparing these results with the EU15 average retirement age (65 years), we find that this optimal retirement is below the statutory value. The minimum deviation goes from 0 to 3 years compared to an average between 7 and 10 years.

A possible explanation of this misalignment is that, while the statutory value is chosen to guarantee the financial sustainability of the pension system at macro level, the optimal value derives from an optimal choice at the micro level.

A second hypothesis relates instead to the growth rate of wages. Indeed, if the wages increase at a lower rate, then it is optimal to retire later. A trend reduction in the average wage growth rate could lead to a slow increase and alignment between the statutory and optimal retirement age.

In our model the ratio between the optimal consumption and the disposable wealth increases over time from 2.5% to about 5.5% and, after retirement, it remains constant over time. Moreover, we find that the percentage of disposable wealth invested in the risky asset is increasing over time, but quite stable in a range between 57% and 58%. Finally, the portfolio volatility is higher at the beginning of the working life and it reaches a minimum at the retirement age.

A Proof of Proposition 1

In solving the first step of Problem (11), we neglect the last term containing only the choice variable T. The Lagrangian of the problem is

$$\begin{aligned}\mathcal{L} :=& \mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\left(\frac{(c_s-c_m)^{1-\delta}}{1-\delta}+\chi_A\frac{(L-l_s)^{1-\delta}}{1-\delta}\mathbb{I}_{s<T}\right)e^{-\rho(s-t_0)-\int_{t_0}^{s}\lambda_u du}ds\right]\\ &+\phi\left(R_{t_0}+\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\left(l_s w_s\mathbb{I}_{s<T}-c_s\right)m_{t_0,s}e^{-r(s-t_0)-\int_{0}^{s}\lambda_u du}ds\right]\right),\end{aligned}$$

where ϕ is the (constant) Lagrangian multiplier. The F.O.C. on consumption for any time s is:

$$(c_s-c_m)^{-\delta}e^{-\rho(s-t_0)-\int_{t_0}^{s}\lambda_u du}-\phi m_{t_0,s}e^{-r(s-t_0)-\int_{t_0}^{s}\lambda_u du}=0,$$

from which

$$c_s^*=c_m+\left(\phi m_{t_0,s}\frac{e^{-r(s-t_0)}}{e^{-\rho(s-t_0)}}\right)^{-\frac{1}{\delta}},$$

while the F.O.C. on labour for any time s is:

$$l_s^*=L-\left(\phi\frac{w_s}{\chi_A}m_{t_0,s}\frac{e^{-(s-t_0)r}}{e^{-\rho(s-t_0)}}\right)^{-\frac{1}{\delta}}.$$

Once c_s^* and l_s^* are substituted into the constraint, rewritten at any time t, we have

$$R_t+\mathbb{E}_t\left[\int_{t}^{\infty}\left(l_s^* w_s\mathbb{I}_{s<T}-c_s^*\right)m_{t,s}e^{-r(s-t)-\int_{t}^{s}\lambda_u du}ds\right]=0,$$

or

$$0 = R_t - \mathbb{E}_t\left[\int_t^{\infty} \left(c_m - Lw_s \mathbb{I}_{s<T}\right) m_{t,s} e^{-r(s-t)-\int_t^s \lambda_u du} ds\right]$$
$$- \phi^{-\frac{1}{\delta}} \mathbb{E}_t\left[\int_t^{\infty} \left(\chi_A^{\frac{1}{\delta}} w_s^{1-\frac{1}{\delta}} \mathbb{I}_{s<T} + 1\right) \left(m_{t_0,s} \frac{e^{-r(s-t_0)}}{e^{-\rho(s-t_0)}}\right)^{-\frac{1}{\delta}} m_{t,s} e^{-r(s-t)-\int_t^s \lambda_u du} ds\right].$$

Now we use the property of the price kernel $m_{t_0,s} = m_{t_0,t} m_{t,s}$ for all $t_0 \leq t \leq s$ and we write

$$\left(m_{t_0,s} \frac{e^{-r(s-t_0)}}{e^{-\rho(s-t_0)}}\right)^{-\frac{1}{\delta}} = \left(m_{t_0,t} \frac{e^{-r(t-t_0)}}{e^{-\rho(t-t_0)}} m_{t,s} \frac{e^{-r(s-t)}}{e^{-\rho(s-t)}}\right)^{-\frac{1}{\delta}},$$

so that the previous equation can be simplified as follows

$$0 = R_t - \mathbb{E}_t\left[\int_t^{\infty} \left(c_m - Lw_s \mathbb{I}_{s<T}\right) m_{t,s} e^{-r(s-t)-\int_t^s \lambda_u du} ds\right]$$
$$- \phi^{-\frac{1}{\delta}} \left(m_{t_0,t} \frac{e^{-r(t-t_0)}}{e^{-\rho(t-t_0)}}\right)^{-\frac{1}{\delta}} \mathbb{E}_t\left[\int_t^{\infty} \left(\chi_A^{\frac{1}{\delta}} w_s^{1-\frac{1}{\delta}} \mathbb{I}_{s<T} + 1\right) m_{t,s}^{1-\frac{1}{\delta}} e^{-\left(\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho\right)(s-t)-\int_t^s \lambda_u du} ds\right].$$

The power $m_{t,s}^{1-\frac{1}{\delta}}$ is not a martingale, and, accordingly, we cannot use it for change the probability. Nevertheless, the process $m_{t,s}^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2(s-t)}$ is a martingale. In fact, given the dynamics (4) we have

$$\frac{d\left(m_{t,s}^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2(s-t)}\right)}{m_{t,s}^{1-\frac{1}{\delta}} e^{\frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2(s-t)}} = -\frac{\delta-1}{\delta}\xi dW_t.$$

Accordingly, we can use Girsanov's theorem for defining a new probability $\mathbb{Q}_\delta$ such that

$$dW_t^{\mathbb{Q}_\delta} = \frac{\delta-1}{\delta}\xi dt + dW_t.$$

Finally, for simplifying the notation we can define the following functions:

$$H_t := \mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{\infty} \left(c_m - Lw_s \mathbb{I}_{s<T}\right) e^{-r(s-t)-\int_t^s \lambda_u du} ds\right],$$

$$F_t := \mathbb{E}_t^{\mathbb{Q}_\delta}\left[\int_t^{\infty} \left(\chi_A^{\frac{1}{\delta}} w_s^{1-\frac{1}{\delta}} \mathbb{I}_{s<T} + 1\right) e^{-\left(\frac{\delta-1}{\delta} r + \frac{1}{\delta}\rho + \frac{1}{2}\frac{1}{\delta}\frac{\delta-1}{\delta}\xi^2\right)(s-t)-\int_t^s \lambda_u du} ds\right],$$

and so

$$R_t = H_t + \phi^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}} \left(\frac{e^{-r(t-t_0)}}{e^{-\rho(t-t_0)}}\right)^{-\frac{1}{\delta}} F_t. \tag{20}$$

The value of the Lagrangian multiplier must be found in such a way that it satisfies the constraint:

$$\begin{aligned}R_{t_0} =&\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\left(c_s^* - l_s^* w_s \mathbb{I}_{s<T}\right) m_{t_0,s} e^{-r(s-t_0)-\int_{t_0}^{s}\lambda_u du} ds\right]\\ =&\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty}\left(c_m - L w_s \mathbb{I}_{s<T}\right) m_{t_0,s} e^{-r(s-t_0)-\int_{t_0}^{s}\lambda_u du} ds\right]\\ &+\phi^{-\frac{1}{\delta}}\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} m_{t_0,s}^{1-\frac{1}{\delta}}\left(1+\chi_A^{\frac{1}{\delta}} w_s^{1-\frac{1}{\delta}}\mathbb{I}_{s<T}\right) e^{-\left(\frac{\delta-1}{\delta}r+\frac{1}{\delta}\rho\right)(s-t_0)-\int_{t_0}^{s}\lambda_u du} ds\right].\end{aligned}$$

Thus, we see that

$$R_{t_0} = H_{t_0} + \phi^{-\frac{1}{\delta}} F_t,$$

from which

$$\phi = \left(\frac{R_{t_0} - H_{t_0}}{F_{t_0}}\right)^{-\delta}.$$

This means that (20) can be written as a function of the initial wealth as follows:

$$\frac{R_t - H_t}{F_t} = \frac{R_{t_0} - H_{t_0}}{F_{t_0}} m_{t_0,t}^{-\frac{1}{\delta}}\left(\frac{e^{-r(t-t_0)}}{e^{-\rho(t-t_0)}}\right)^{-\frac{1}{\delta}}.$$

Given the stochastic differential equations for the wage (5) and for the price kernel (4), the differential of R_t is

$$\begin{aligned}dR_t =& (\ldots)\, dt + \left(\frac{\partial H_t}{\partial w_t} + \phi^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}\left(\frac{e^{-r(t-t_0)}}{e^{-(t-t_0)\rho}}\right)^{-\frac{1}{\delta}}\frac{\partial F_t}{\partial w_t}\right)\sigma_w dW_t\\ &-\left(-\frac{1}{\delta}\phi^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}-1}\left(\frac{e^{-r(t-t_0)}}{e^{-(t-t_0)\rho}}\right)^{-\frac{1}{\delta}} F_t\right) m_{t_0,t}\xi dW_t,\end{aligned}$$

and, since

$$\frac{R_t - H_t}{F_t} = \phi^{-\frac{1}{\delta}} m_{t_0,t}^{-\frac{1}{\delta}}\left(\frac{e^{-r(t-t_0)}}{e^{-(t-t_0)\rho}}\right)^{-\frac{1}{\delta}},$$

it becomes

$$dR_t = (\ldots)\, dt + \left(\frac{\partial H_t}{\partial w_t} + \frac{R_t - H_t}{F_t}\frac{\partial F_t}{\partial w_t}\right)\sigma_w dW_t + \frac{R_t - H_t}{\delta}\xi dW_t.$$

The optimal portfolio is then

$$S_t w_t^* = \frac{R_t - H_t}{\delta}\frac{\xi}{\sigma} + \frac{\sigma_w}{\sigma}\left(\frac{\partial H_t}{\partial w_t} + \frac{R_t - H_t}{F_t}\frac{\partial F_t}{\partial w_t}\right),$$

while the optimal consumption and labour are those presented in the proposition.

B Proof of Proposition 3

Given a geometric Brownian motion

$$\frac{dX_t}{X_t} = \alpha dt + \beta dW_t,$$

whose solution is (with $W_{t_0} = 0$)

$$X_t = X_{t_0} e^{\left(\alpha - \frac{1}{2}\beta^2\right)(t-t_0)+\beta W_t},$$

we want to compute the expected value $\mathbb{E}_{t_0}\left[\int_{t_0}^{T} X_s e^{-\rho(s-t_0)} ds\right]$ that can be simplified as follows

$$\begin{aligned}
&\mathbb{E}_{t_0}\left[\int_{t_0}^{T} X_s e^{-\rho(s-t_0)} ds\right] \\
=&\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} X_s e^{-\rho(s-t_0)} ds - \int_{T}^{\infty} X_s e^{-\rho(s-t_0)} ds\right] \\
=&\mathbb{E}_{t_0}\left[\int_{t_0}^{\infty} X_s e^{-\rho(s-t_0)} ds\right] - \mathbb{E}_{t_0}\left[\int_{T}^{\infty} X_s e^{-\rho(s-t_0)} ds\right] \\
=&\int_{t_0}^{\infty} \mathbb{E}_{t_0}\left[X_s\right] e^{-\rho(s-t_0)} ds - \mathbb{E}_{t_0}\left[e^{-\rho T}\int_{T}^{\infty} X_s e^{-\rho(s-T)} ds\right] \\
=&\int_{0}^{\infty} X_{t_0} e^{-(\rho-\alpha)s} ds - \mathbb{E}_{0}\left[e^{-\rho T}\int_{T}^{\infty} \mathbb{E}_T\left[X_s\right] e^{-\rho(s-T)} ds\right] \\
=&\frac{X_{t_0}}{\rho-\alpha} - \mathbb{E}_{t_0}\left[e^{-\rho T} X_T \int_{T}^{\infty} e^{-(\rho-\alpha)(s-T)} ds\right] \\
=&\frac{X_{t_0}}{\rho-\alpha} - \mathbb{E}_{t_0}\left[e^{-\rho T} \frac{X_T}{\rho-\alpha}\right] \\
=&\frac{X_{t_0} - \mathbb{E}_{t_0}\left[X_T e^{-\rho T}\right]}{\rho-\alpha}.
\end{aligned}$$

Then, it is now sufficient to compute the remaining expected value $\mathbb{E}_{t_0}\left[X_T e^{-\rho T}\right]$. If we call x the value of the process at the stopping time T (i.e. $x = X_T$), then

$$\mathbb{E}_{t_0}\left[X_T e^{-\rho T}\right] = x^{1-\gamma} X_{t_0}^{\gamma},$$

where

$$\gamma = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + 2\frac{\rho}{\sigma^2}}.$$

Finally, we can write

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{T} X_s e^{-\rho(s-t_0)} ds\right] = \frac{X_{t_0} - x^{1-\gamma} X_{t_0}^{\gamma}}{\rho-\alpha}.$$

These same results allow us to write also

$$\mathbb{E}_{t_0}\left[\int_{t_0}^{T} e^{-\rho(s-t_0)}ds\right] = \frac{1-\mathbb{E}_{t_0}\left[e^{-\rho T}\right]}{\rho} = \frac{1-x^{-\gamma}X_{t_0}^{\gamma}}{\rho}.$$

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Fondazione Eni Enrico Mattei
Corso Magenta 63, Milano – Italia

Tel. +39 02.520.36934
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