QUANTIFYING LONG RUN AGRICULTURAL RISKS AND EVALUATING FARMER RESPONSES TO RISK

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MODELING NONNORMALITY IN MULTIVARIATE DISTRIBUTIONS USING AN INVERSE HYPERBOLIC SINE TRANSFORMATION TO NORMALITY

by Charles B. Moss, Octavio Ramirez and William G. Boggess*

During the past twenty years there has been a growing recognition of the consequences of the randomness of crop yields and prices for farm management and agricultural policy decisions. Concomitantly, the literature has recognized the possibility of nonnormality of yields and prices. This study demonstrates one approach to modeling nonnormality in crop yields over time. Specifically, this study estimates an inverse hyperbolic sine transformation of crop yields in the southeastern United States.

keywords: nonnormality, crop yields, inverse hyperbolic sine

Several questions in the areas of farm management and agricultural policy have arisen over the past twenty years that require knowledge of the distribution of random variables. At the farm level the distribution of crop yields and prices may affect the choice of crop portfolio, financial decisions, or participation in government programs. Some of these decisions have been incorporated into ARMS (King). At the policy level, the distribution of crop yields may affect the outlay on government programs, farm survival, and even agricultural trade policy. One such application is the farm level simulator FLIPSIM (Richardson and Nixon).

Certain characteristics of the distribution of crop yields may be particularly important in these models. In the past, agricultural economists

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*Charles Moss is an Assistant Professor of Food and Resource Economics at the University of Florida, Octavio Ramirez is an Assistant professor at ENCATI. William Boggess is a Professor of Food and Resource Economics at the University of Florida.

have primarily focused on in the mean and variance of yield distributions. The mean of the distribution has been used both as a deterministic measure of profitability and as an element in more complex analysis of risk. The variance is often used as a measure of risk following the basic formulation of Freund. However, several studies have examined the possibility that crop yields are nonnormal. Day found that yields on crops in the Mississippi delta region were skewed and kurtotic. Similarly, Gallagher found that soybean yields were negatively skewed. However, Luttrell and Gilbert rejected the hypothesis that crop yields were "bunchy" due to weather.

The potential of nonnormal yield distributions may be of particular importance in farm management and agricultural policy problems. If the yield distribution is skewed financial decisions, crop insurance models, and many farm policies such as target prices may misstate the value of liquidity, the actuarial cost and value of insurance, or the potential cost of farm programs, respectively. Complicating the issue of nonnormality is the possibility of correlation or interdependence. Agricultural yields tend to move together because of weather and pests. A below normal winter rainfall decreases both wheat and barley yields. Further, low crop yields over a substantial area can lead to higher crop prices. Interdependence in crop yields and prices could have a significant affect on farm management and agricultural policy decisions.

A methodology for modeling correlated random variables under normality has been presented by Clements et al. However, modeling correlated random variables given skewness or kurtosis is far more complicated. King and Richardson and Condra have presented methods for modeling correlation given nonnormality. However, these approaches are not based on a theoretically specified multivariate distribution. Taylor has proposed methodology to model correlated nonnormal variates, but his approach is sensitive to ordering.

The purpose of this paper is to present a methodology for estimating and simulating multivariate distributions in a manner that allows for nonnormality. The proposed methodology is based on estimating the parameters of an inverse hyperbolic sine transformation into normality. This transformation allows normality as a special case. The methodology is then used to estimate crop yield distributions in the southeastern United States.

Transforming Random Variables Using the Inverse Hyperbolic Sine

The inverse hyperbolic sine transformation was suggested by Johnson and applied by Burbidge et al. The multivariate form applied in this manuscript was adapted by Ramirez. The basic concept is to fit the parameters that transform a random variable into another distribution with desirable properties. A similar transformation is the Box-Cox transformation. In the current scenario we are interested in transforming a random variable in a way that allows for nonnormality, but has an explicit joint distribution function. The inverse hyperbolic sine distribution models the nonnormality of the marginal distribution. This transformation maps the nonnormal variables into a joint normal distribution which allows for contemporaneous interdependence.

A basic concept from mathematical statistics is that functions of random variables possess a distribution defined by the distributions of the original random variables. Examples of this property include several distributions used for classical statistical tests such as the Student's t. In this paper we are particularly interested in one of the procedures used to define these distributions. Specifically, Mood, Graybill, and Bowes show that a monotonic transformation $y=g(x)$ can be used to define a distribution for y based on the distribution of x,

$$(1) \qquad f_y(y) = \left| \frac{dg^{-1}(y)}{dy} \right| f_x(g^{-1}(y))$$

where $f_y(y)$ is the marginal distribution of y, $g^{-1}(y)$ is the inverse mapping based on $y=g(x)$, and $f_x(x)$ is the distribution of x. Thus, given a monotonic transformation $g(x)$ we can define $f_y(y)$ based on $f_x(x)$.

This study proposes using the inverse hyperbolic sine to transform crop yields. The univariate inverse hyperbolic sine distribution presented in Burbidge et al. is

$$(2) \qquad g(y_t,\theta) = \frac{\ln(\theta y_t + (\theta y_t)^2 + 1)^{\frac{1}{2}})}{\theta}$$

where y_t is the observed variable and θ is a parameter that controls kurtosis. Ramirez made two basic modifications to the univariate mapping. First, Ramirez suggested that y_t be replaced with e_t. Hence, the errors could be nonnormal apart from any deterministic component. Second, Ramirez added a centrality parameter μ. The centrality parameter in conjunction with θ can be used to control skewness. Thus, the univariate form of the transformation estimated in this study becomes

$$(3) \qquad \begin{aligned} v_t &= y_t - \hat{y}_t \\ z_t &= \frac{\ln(\theta v_t + ((\theta v_t)^2 + 1)^{\frac{1}{2}})}{\theta} \\ e_t &= z_t - \mu \end{aligned}$$

where v_t is the deviation from the deterministic model, $\hat{y}_t$ is the prediction, z_t is the transformed deviation, e_t is the normally distributed deviation, and μ is the centrality constant.

The nature of this transformation and its effect on the probability density functions are shown in Figures 1 and 2 respectively. In Figure 1 the straight line depicts the "normal" transformation. Specifically, as θ approaches zero, the inverse hyperbolic sine transformation becomes a straight line. If θ is nonzero, negative in the current case, then the transformation becomes s-shaped. The sigmoid shape of the curve is symmetric with respect to the origin. The effect of this transformation is to change the kurtosis of the distribution. Finally, μ shifts the curve away from the origin inducing nonsymmetry. This nonsymmetry manifests itself as skewness.

In Figure 2 the base graph represents the deviation from trend for corn in the southeast assuming a normal density function (i.e. $\theta = \mu = 0$). The symmetric distribution around the normal distribution function represents the change in the distribution when only the kurtosis is allowed to change (i.e. $\theta < 0$, $\mu = 0$). Note the difference between the kurtotic distribution and a simple change in variance. Specifically, the probability of a zero draw increases as does the probability of a draw at -20. Hence, the distribution has not simply been flattened, rather the relationship between the second and fourth moments has been changed. The effect of skewness (i.e. $\theta < 0$, $\mu < 0$) in this case is far more dramatic as illustrated by the negatively skewed density function in Figure 2. Thus, the transformation using the inverse hyperbolic sine allows both skewness and kurtosis to be modeled.

The transformation can be extended to multivariate space following the transformation results presented in Hogg and Craig

$$(4) \qquad f_y(y) = f_x(g^{-1}(y)) \begin{vmatrix} \frac{\partial g_1^{-1}(y)}{\partial y_1} & \cdots & \frac{\partial g_1^{-1}(y)}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial g_m^{-1}(y)}{\partial y_1} & \cdots & \frac{\partial g_m^{-1}(y)}{\partial y_n} \end{vmatrix}$$

where $g^{-1}(y)$ is a vector inverse mapping function and $g_i^{-1}(y)$ is the inverse mapping function for the ith element from that mapping function. Following Ramirez, the jacobian matrix of the transformation is assumed to be diagonal,

(5)
$$\frac{\partial g_i^{-1}(y)}{\partial y_j}=(1+(\theta_i y_i)^2)^{-\frac{1}{2}} \quad i=j$$
$$\frac{\partial g_i^{-1}(y)}{\partial y_j}=0 \quad for \quad i\neq j$$

therefore, the determinant of the jacobian is the product of the diagonal elements. This implies that the marginal transformations are independent. The multivariate density function can then be expressed as

(6)
$$f_{v_t}=\frac{1}{\sqrt{2\pi}}|\Omega|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(z_t-\mu)\Omega^{-1}(z_t-\mu)'\right)\prod_{i=1}^{M}(1+(v_{it}\theta_i)^2)^{-\frac{1}{2}}$$
$$z_t=\begin{bmatrix}\ln(v_{1t}\theta_1+((v_{1t}\theta_1)^2+1)^{\frac{1}{2}})/\theta_1\\ \ln(v_{2t}\theta_2+((v_{2t}\theta_2)^2+1)^{\frac{1}{2}})/\theta_2\\ \vdots\\ \ln(v_{mt}\theta_m+((v_{mt}\theta_m)^2+1)^{\frac{1}{2}})/\theta_m\end{bmatrix}$$

where Ω is the variance matrix of the transformed residuals, z_t is a vector of transformed residuals, μ is the vector of noncentrality parameter and θ is the vector of kurtosis parameters.

Estimation of the Inverse Hyperbolic Sine System

Using the probability density function in equation (6), a maximum likelihood estimation procedure can be employed to estimated the parameters depicting yield distributions over time. We present a model where yields follow a deterministic linear trend with nonnormal deviations. This procedure jointly estimates the parameters of the linear trend and the nonnormal transformation. This departs from other studies that estimate the time trend using ordinary least squares, then estimate the higher moments using the residuals from that regression. In addition, the suggested procedure is sequential by necessity, because the iterative procedures used in the maximum likelihood estimation require starting values which cannot be zero in the case of the kurtosis parameter.

The first step in the proposed procedure is to test the individual series for nonnormality. Hence, we use ordinary least squares to derive estimated deviations. These deviations are then tested for normality using the skewness, kurtosis, and joint skewness and kurtosis tests outlined by Harvey. If either skewness or kurtosis is found, the next step is to fit an individual inverse hyperbolic sine representation. These univariate representations are used as starting values for the joint estimation process described later. If the deviations cannot be distinguished from normality, the distribution is assumed to be normal (i.e. $\theta=\mu=0$).

To estimate the nonnormality parameters the ordinary least squares estimates of trend and standard error are used as initial estimates. From the ordinary least squares results in the first step we obtain

$$(7) \qquad \hat{b}=(x'x)^{-1}(x'y)$$
$$\hat{e}=y-x\hat{b}$$

where $\hat{b}$ are the estimated trend parameters, x is a matrix consisting of a column of ones and a linear trend column, y is the observed level of yields, and $\hat{e}$ are the estimated residuals. Maximum likelihood can then be used to fit θ and μ based on these residuals

$$(8) \qquad \underset{\theta,\mu,\sigma^2}{Max} \quad L_1=-\frac{T}{2}\ln\sigma^2-\frac{1}{2}\sum_{t=1}^{T}\frac{(z_t-\mu)^2}{\sigma^2}-\frac{1}{2}\sum_{t=1}^{T}\ln(1+(\hat{e}_t\theta)^2)$$
$$z_t=\ln(\hat{e}_t\theta+((\hat{e}_t\theta)^2+1)^{\frac{1}{2}})/\theta$$

One problem with this approach is that the estimated deviations are based on the estimated trend parameters which may be biased by construction. Ordinary least squares is an efficient estimator given that the errors of the regression are normally distributed. However, by the very nature of the problem the deviations are assumed nonnormal. Hence, the ordinary least square estimator of the regression parameters may be inappropriate. To overcome this problem, the results of equation (8) are used as starting values in the likelihood function

$$(9) \quad \max_{\alpha_0,\alpha_1,\theta,\mu,\sigma^2} L_2=-\frac{T}{2}\ln(\sigma^2)-\frac{1}{2}\sum_{t=1}^{T}\frac{(z_t-\mu)^2}{\sigma^2}-\frac{1}{2}\sum_{t=1}^{T}\ln(1+(v_t\theta)^2)$$

$$v_t=y_t-\alpha_0-\alpha_1 t$$

$$z_t=\ln(\theta v_t+((\theta v_t)^2+1)^{\frac{1}{2}})/\theta$$

These results can be compared with the ordinary least squares likelihood function to determine the statistical significance of the transformation. Specifically, the difference between L_2 and the log likelihood function of the ordinary least squared regressor is distributed χ^2 with two degrees of freedom.

The final step is to use the individual maximum likelihood results as starting points to estimate the system. The likelihood function used in this step is simply the log of equation (6),

$$(10) \quad \max_{\alpha_0,\alpha_1,\theta,\mu,\Sigma} L_3=-\frac{T}{2}\ln|\Omega|-\frac{1}{2}\sum_{t=1}^{T}(z_t-\mu)\Omega^{-1}(z_t-\mu)' -\frac{1}{2}\sum_{t=1}^{T}\sum_{i=1}^{m}\ln(1+(v_{it}\theta)^2)$$

$$v_{it}=y_{it}-\alpha_{i0}-\alpha_{i1}t \quad i=1,\ldots m$$

$$z_{it}=\ln(v_{it}\theta_i+((v_{it}\theta_i)^2+1)^{\frac{1}{2}})/\theta_i.$$

From an applications standpoint, this equation maybe difficult to maximize.[1] As efficient procedure is to begin with a steepest descent algorithm and then to switch to Newton-Raphson when the Hessian becomes well behaved. In addition, initial off diagonal estimates of the covariance matrix can be obtained by transforming the residuals into normality. Also, this estimation can be

[1]By construction of the inverse hyperbolic sine transformation, skewness in the distribution is jointly determined by θ and μ. Because kurtosis uniquely modeled by θ, the occurrence of skewness without significant kurtosis can complicate estimation. This is a primary reason behind our suggestion of steepest descent algorithms over more efficient Newton-Raphson algorithms.

simplified if one or more of the variables are normally distributed. For example, if the first variable is normal then θ_1 approaches zero, and

$$z_{it}=\ln(v_{it}\theta_i+((v_{it}\theta_i)^2+1)^{\frac{1}{2}})/\theta_1$$

becomes z_{it} = vit and $\ln(1+(v_{it}\theta_i)^2) = \ln 1 = 0$.

Estimated Results

The inverse hyperbolic sine transformation was estimated for wheat, corn, oats, and barley in the southeastern United States using average regional yields from 1950-1989. A deterministic linear trend is applied and the deviations from this trend are allowed to be nonnormally distributed.

Initial tests for normality presented in Table 1 indicate that normality cannot be rejected for wheat yields. However, corn, oats and barley appear to be skewed, but none of these distributions exhibit kurtosis. The joint hypothesis for both skewness and kurtosis can be also rejected for corn and oats. Based on these results, univariate inverse hyperbolic sine distribution were fit for corn, oats and barley.

Next, the univariate inverse hyperbolic sine transformation for corn, oats and barley in the southeastern United States from equation (9) was estimated using maximum likelihood. These results are presented in Table 2. As anticipated, the results from the inverse hyperbolic sine estimation have relatively large skewness parameters and small kurtosis parameters. Using these values as starting points, the full information system in equation (10) was estimated. These results are presented in Table 3.

The estimates for the transformation variables from the full information maximum likelihood are close to the limited information maximum likelihood for corn and barley. However, the additional information caused significant changes in the transformation for oats. In addition, the results from the full information maximum likelihood possess both intercepts and deterministic trend parameter estimates for all variables. The intercept of the deterministic trend (α_0) increased substantially for wheat, oats and barley. However, the annual increase (α_1) is relatively close to the single equation estimates.

The "variance" matrix presented in Table 3 requires additional elaboration. Technically, the variance matrix presented in Table 3 is the covariance matrix for the transformed random variables. Hence, it cannot be interpreted without the transformation. This is of particular importance in the results for oats. According to the Table 3, the current variance of oat production is .0171 which is very low. However, the transformation parameters for this equation are now relatively large. Thus, it would be incorrect to interpret this parameter as the variability in oat yields. The appropriate variance can be computed utilizing the moment generating functions from Ramirez.

Simulating Nonnormal Variables

The major focus of this paper is to model nonnormal correlated random variables. The basis for the correlated portion of this objective is the variance matrix of the transformed random variables. The nonnormality is modeled using the inverse hyperbolic sine transformation.

The first step in risk simulation is to generate a set of normal correlated random draws using the transformed variance matrix estimates from Table 3. Following Clements et al. a matrix (Tx4) of N(0,1) variates are drawn using a psuedo random number generator. This vector is then multiplied by the Choleski decomposition of the transformed variance matrix. This results in a matrix of correlated normal deviates which are then transformed using the hyperbolic sine. Specifically,

$$y=\sinh(z\theta)/\theta$$

$$(11)\quad \frac{\sinh(z\theta)}{\theta}=\left\{\begin{array}{c} z \text{ if } \theta=0 \\ \frac{e^{\theta z}-e^{-\theta z}}{\theta} \text{ if } \theta\neq 0 \end{array}\right\}$$

Using this procedure, the estimated parameters in Table 3 were used to generate 750 samples of wheat, corn, oats and barley in the southeast region. Figure 3 presents the histogram for wheat yields in this simulated sample. Consistent with the formulation, wheat yields appear to be normally distributed around a mean of 36.56 bushels. Figures 4-6 give the histograms for corn, oats and barley respectively. The average simulated yield for corn was 82.95 bushels per acre, the average simulated yield for oats was 60.14 bushels per acre, and the average simulated yield for barley was 54.61 bushels per acre. In all three cases the distributions are negatively skewed. Hence, the simulated distributions conform with the estimated parameters.

Conclusions and Discussion

Nonnormal correlated random events can plan an important role in several areas of agricultural economics research. Farm management and agricultural policy research may be particularly interested in correlated nonnormal random draws since there is evidence that crop yields and prices are often correlated and may be kurtotic and skewed. These properties may have profound affects on firm decisions and, hence, on the costs and benefits of agricultural policy.

This study presented one approach to modeling nonnormality which allows for correlation. Specifically, this study estimated an inverse hyperbolic sine transformation into normality for wheat, corn, oats and barley yields in the southeastern United States. The estimated parameters where then used to simulate crop yields.

The methodology presented allows for the estimation of nonnormal correlated random variates based on a transformation to normality. This transformation has several desirable advantage over previous approaches. First, the inverse hyperbolic sine transformation to normality has an explicit interaction term, the transformed variance matrix. Hence, the interdependence between random variables is an explicit part of the density function. Second, normality using nonnormality is a special case of the transformation. Further, it is possible to test for normality using the estimated parameters. Third, the complete distribution function is known. Thus, the interaction terms and transformation parameters can be estimated at the same time. Hence, the ordering of the variables makes no difference unlike conditional approaches. Fourth, this formulation allows the joint estimation of the trend parameters.

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Table 1: Preliminary Tests for Nonnormality of Deviations from Trend Yields

Yield	Significance of Test[a]		
	Skewness	Kurtosis	Both Skewness and Kurtosis
Wheat	.2830	.8643	.6651
Corn	.0000[b]	.7027	.1149
Oats	.0000[b]	.9996	.1018
Barley	.0000[b]	.9998	.0022

[a]Test values are the observed significance levels of the hypothesis that the deviations conform to normality. The test for normality are taken from Harvey.

[b]Significance level rounds to zero.

Table 2: Estimates of the Univariate Inverse Hyperbolic Sine Transformation Parameters

Equation	Parameter Estimates				
	θ	μ	α_0	α_1	σ^2
Wheat	-	-	14.2376	.7857	17.7384
Corn	-.1527	-7.2803	21.6699	1.8010	17.9287
Oats	-.2433	-2.0414	28.5714	.8347	9.1978
Barley	-.1744	-2.0374	25.5142	.7580	17.1251

Table 3: Estimates of Distributional Parameters Using Full Information								
	Parameter Estimates							
Equation	θ	μ	α_0	α_1	ω^a_{Wheat}	ω_{Corn}	ω_{Oats}	ω_{Barley}
Wheat	-	-	18.0063	.4581	17.8989			
Corn	-.1368	-8.0571	21,8733	1.7774	-5.1161	20.7193		
Oats	-3.0117	-1.3844	36.9516	.85338	.0336	-.0152	.0171	
Barley	-.1372	-10.3006	36.7687	.80753	6.3690	-.5145	.2658	6.3397

[a]The estimated parameters ω are the lower triangle of the Ω matrix which represents the variance matrix of the transformed residuals. In the case of ω_{wheat}, this parameter is the variance of wheat yields.

Figure 1: Inverse Hyperbolic Sine Transformation

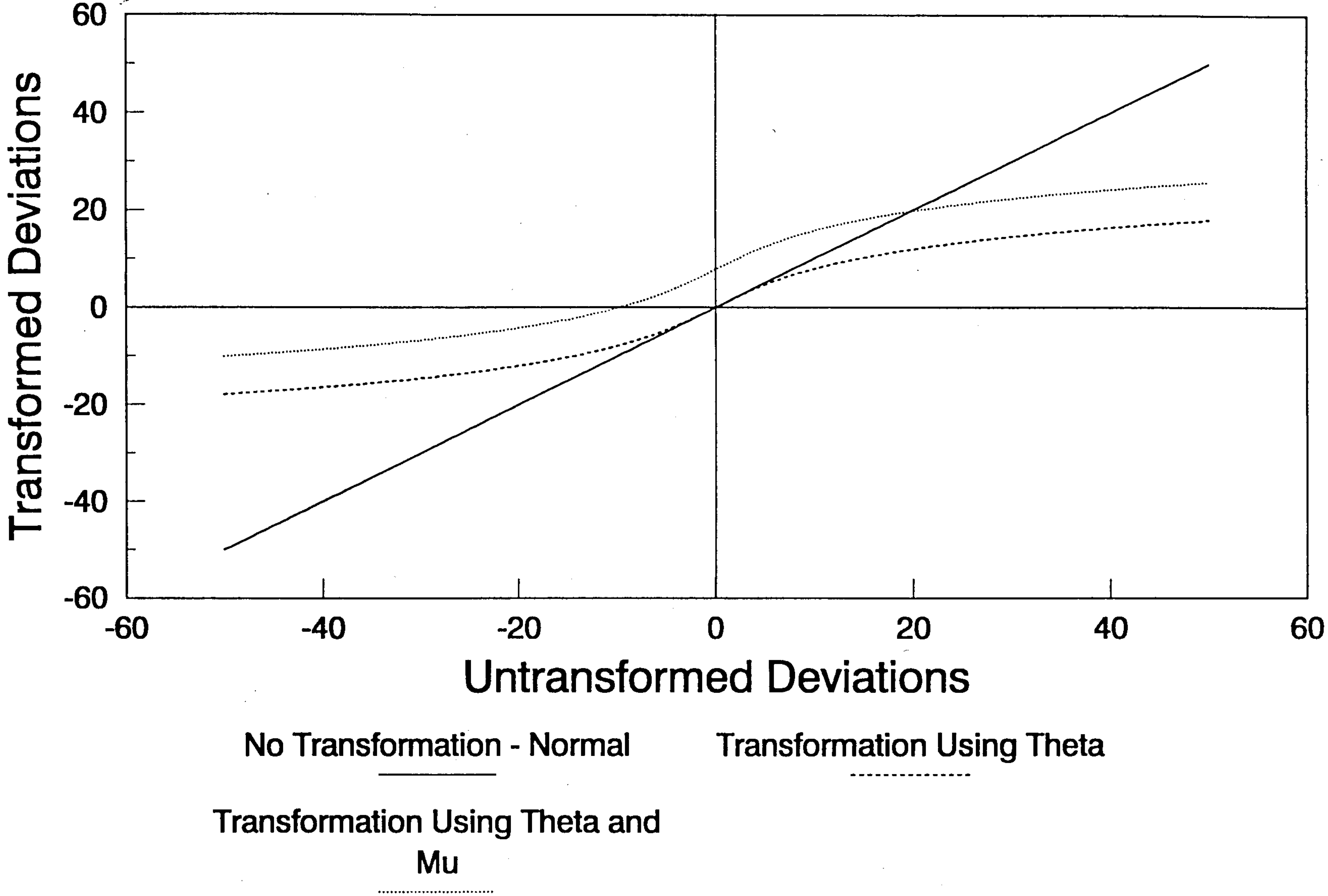

Figure 2: Alternative Density Functions

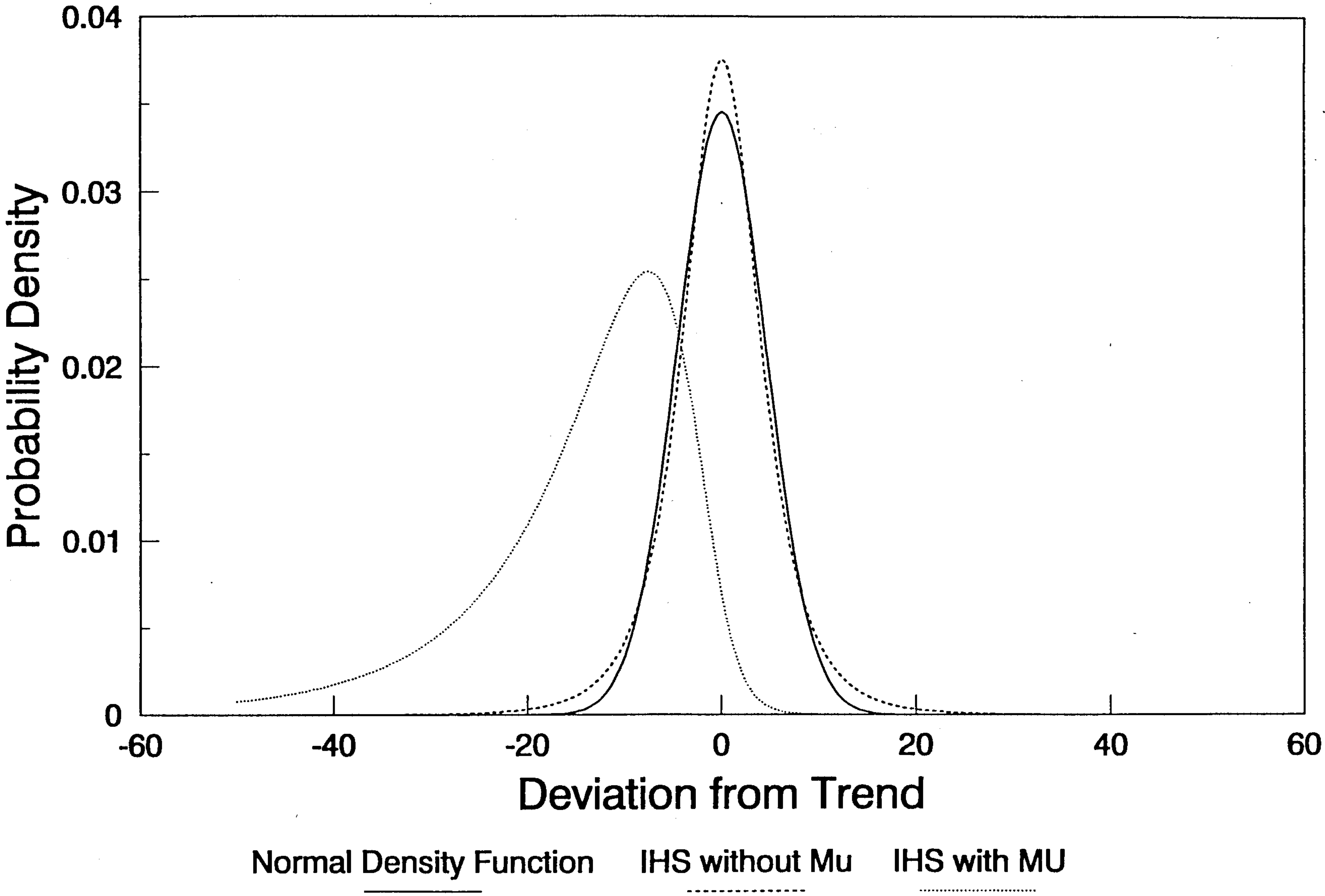

Figure 3: Simulated Wheat Distribution

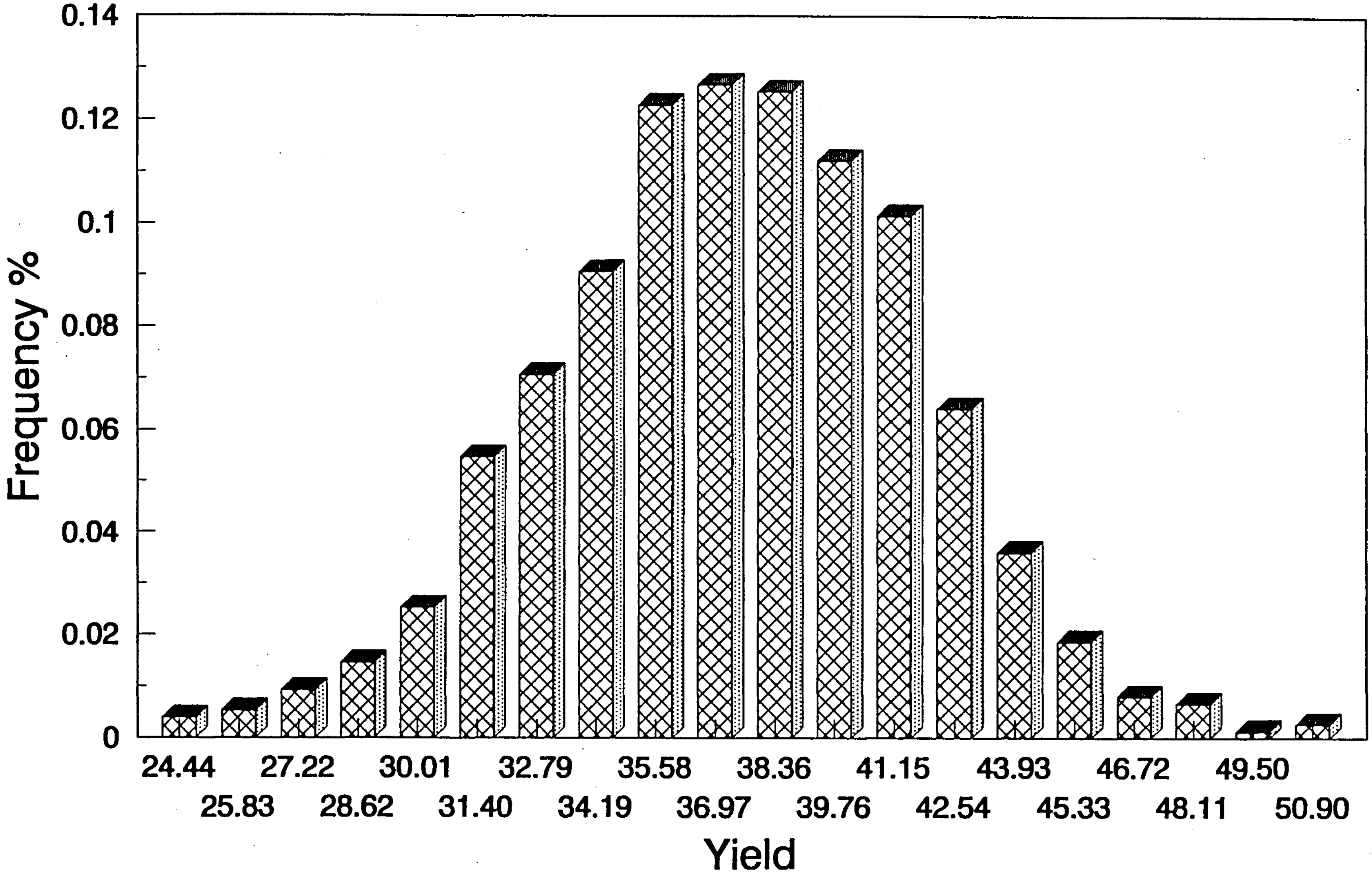

Figure 4: Simulated Corn Yield

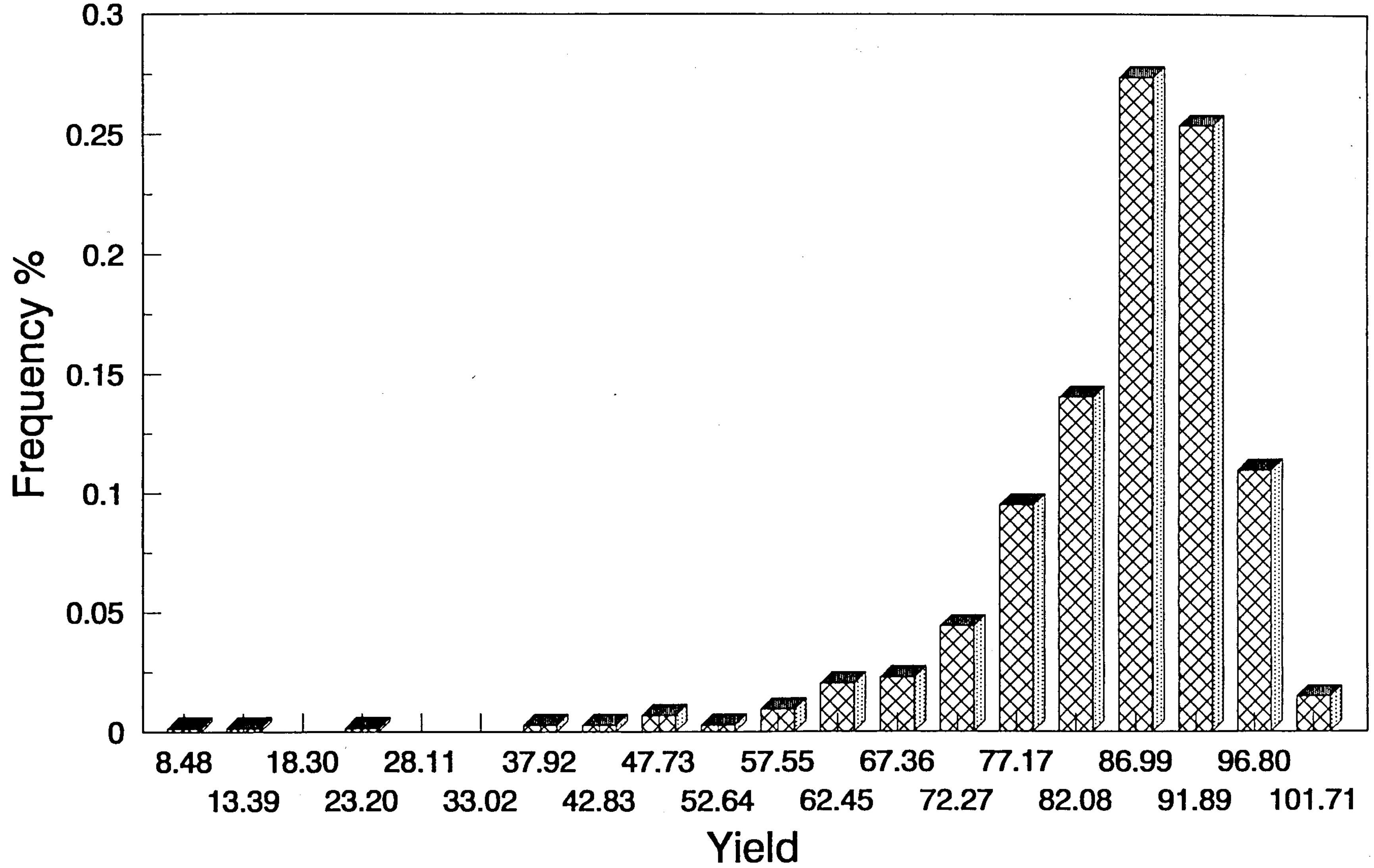

Figure 5: Simulated Oat Yields

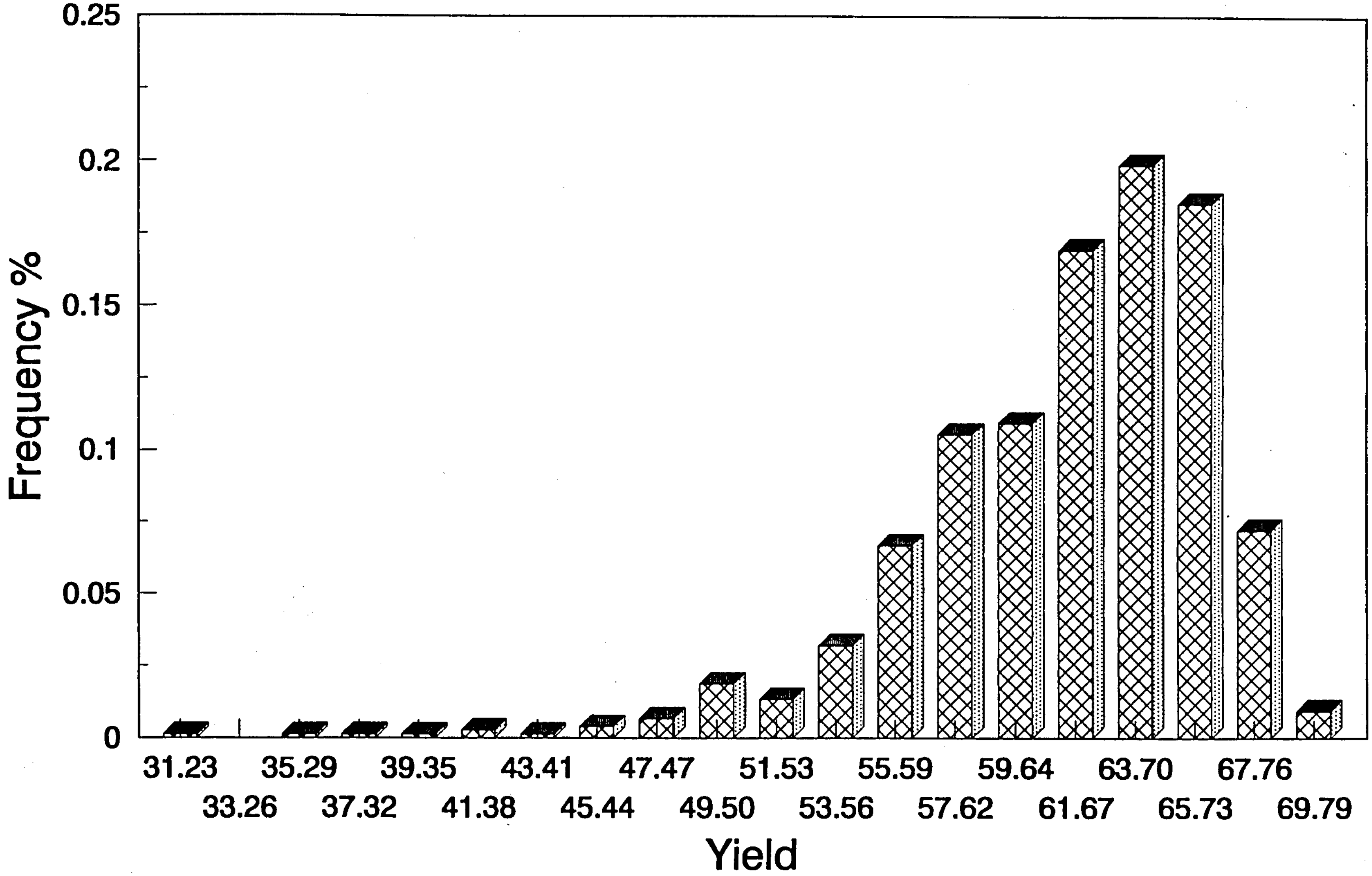

Figure 6: Simulated Barley Yields