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Slutsky, Let Me Introduce You to Arrow-Pratt: Competitive Price Effects with Uncertain Production

by

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ABSTRACT: The purpose of this article is to characterize the effect of a competitive price change on a producer's commodity transactions under uncertainty and impatience. The novelty comes from a methodological approach inspired by both Slutsky and Arrow-Pratt. Combining and generalizing these methodological frameworks illuminates natural analogues between production and consumption with and without uncertainty, while facilitating the analysis of certain, risky, and uncertain choice within a consistent framework. Contributions include (i) the introduction of the immediate profit function — a generalization of cost function to an economic environment with uncertainty and impatience, (ii) a generalization of Arrow-Pratt risk aversion to characterize preferences over time as well as over uncertainty, (ii) a generalization of Arrow-Pratt risk aversion to characterize technological and market uncertainty, (iv) the decomposition of price effects on commodity choices with uncertainty and impatience using Slutsky substitution and income effects, and the generalized Arrow-Pratt characterizations of uncertainty aversion, patience aversion, technological uncertainty and market uncertainty; and (v) a reexamination of Sandmo's seminal comparative static analysis of a producer facing price risk in the context of price uncertainty and impatience.

KEYWORDS: state-contingent, production, risk, uncertainty, comparative static

JEL Classification: D21, D8

1 Introduction

Early research exploring risky production assessed the effect of a producer's choices on the distribution of profit when prices or output are random and then exploited expected utility theory to characterize optimal choices in relation to risk attitudes and changes in the production environment (e.g., Baron 1970; Feldstein 1971; Sandmo 1971; Leland 1972; and Ratti and Ullah 1976). These efforts produced analytically tractable and intuitively appealing results, which encouraged widespread adoption of what can be referred to as random profit models. While random profit models helped researchers understand risky production, their development was not well connected to production and consumption theory (PCT) where objects like supply, demand, production possibility frontiers, indifference curves, and isoquants; and taxonomies like income versus substitution effects, substitutes versus complements, and normal versus inferior goods have served as fundamental concepts for instruction, empirics and applied theory. Instead, researchers seeking to contribute to or students trying to master results from this literature had to set aside much of these PCT concepts and open up a new set of analytical tools with objects like lotteries, probability distributions, expected profit and utility, certainty equivalents, risk premiums, mean preserving spreads, and stochastic dominance; and taxonomies like risk seeking versus averse, increasing versus decreasing risk aversion, risk increasing versus decreasing inputs, risk substitutes versus complements, and pure risk versus expansion effects.

The necessity of dispensing with PCT foundations in order to understand risky production has been challenged in the past two decades through the extension and application of Arrow's (1953) and Debreu's (1959) state contingent models of choice under uncertainty (Chambers and Quiggin 1997, 1998, 2000, and 2001; and Quiggin and Chambers 1998 and 2001). Three notable accomplishments of this literature are the establishment of more transparent links between the theory of certain and uncertain production, the extension of duality results from certain to uncertain production, and a reduction in the reliance on expected utility. The rewards of these accomplishments are pedagogical, empirical, and practical. Pedagogically, PCT foundations for risky production make it possible to convey concepts of risk and uncertainty with tools taught to students beginning with first principles. Empirically, there is a vast tool box for analyzing consumption and production decisions in a certain world that could be opened up to analyzing production in an uncertain world. Practically, much in the world is uncertain and evidence continues to mount that interpreting it through an expected utility lens is problematic.

Great progress has been made, but gaps remain. The purpose of this article is to fill one such gap. The gap we have in mind is the lack of a PCT interpretation of competitive price effects for uncertain production. In particular, we develop a Slutsky style equation that decomposes the effect of a price change on commodity transactions into substitution and income effects. Some of these effects are unambiguous, while more common ambiguous results invoke analogs to PCT taxonomies of substitute versus complement and normal versus inferior commodities. The results also invoke analogies to Arrow-Pratt risk aversion, which are shown to generally relate to changes in marginal rates of substitution and transformation in response to changes in the production environment.

Chambers and Quiggin (2001) decomposes the effect of a shock to the economic environment on input choices, but they do so by using pure-risk and expansion effects related to Rothschild and Stiglitz (1970, 1971) notions of mean preserving spreads, and a novel taxonomy of risk complements versus substitutes. While their decomposition is inspired by Slutsky, it is distinct from the PCT foundations used here and draws on notions of risk developed in the context of expected utility that do not have obvious analogues in PCT (e.g., concepts like certainty equivalents, risk premiums, and stochastic dominance make less sense when looking at bundles of goods like food, clothing, transportation and shelter instead of farmer profit in years of normal rainfall, flood or drought). Their approach also relies heavily on probabilistic sophistication (e.g., Machina and Schmeidler 1992) and globally risk averse preferences, the latter of which is inconsistent with empirical regularities such as individuals who are risk averse or seeking depending on whether there is a gain to be had or loss to avoid (Kahneman and Tversky 1979).

The contributions of this article are (i) a generalization of the cost and revenue cost function that we refer to as the immediate profit function, (ii) a generalization of Nau's (2003, 2011) uncertainty aversion matrix and the Arrow-Pratt taxonomy of relative and absolute risk aversion that includes temporal dimensions of preferences that can be applied locally as well as globally, (iii) a generalization of Nau's uncertainty aversion matrix that characterizes technological and market uncertainty either locally or globally, (iv) the decomposition of the effects of a price change on uncertain input and output choices that is interpretable in the context of PCT substitution and income effects as well as our generalizations of the Arrow-Pratt risk aversion coefficients to uncertainty aversion, patience aversion, technological uncertainty and market uncertainty; and (v) a reexamination of Sandmo's (1971) comparative static effects for a producer facing price risk that explores how robust the results are to preferences over time as well as uncertainty.

Section 2 outlines the state contingent framework for our analysis, and characterizes the immediate profit function as its dual representation. Preferences are outlined in Section 3 and conditions for optimal production are derived. Section 4 develops generalizations of Arrow-Pratt risk aversion coefficients for uncertain technology and markets as well as uncertain and impatient preferences. Section 5 develops the general comparative static price effects with Slutsky and Arrow-Pratt interpretations. Section 6 revisits Sandmo's comparative static analysis using Arrow-Pratt interpretations of uncertain and impatient preferences. Section 8 concludes.

2 Technology

There are L distinct commodities that are purchased or sold. These transactions can be primary commodities like land and labor or commodities produced from other commodities like cotton and cloth. The transactions may be certain or uncertain due to a dependence on the realization of some event, which makes it natural to think of certain transactions as immediate while uncertain transactions are negligibly or substantially delayed. If there is substantial delay before uncertainty is resolved, temporal as well as uncertain preferences can become important for decision making. Therefore, we clearly delineate certain transactions, which we refer to as immediate transactions, and uncertain transactions.

Immediate transactions are denoted by the netput vector $\mathbf{y^0} \in \mathbb{R}^L$ where $y_l^0 > (<) 0$ implies the lth commodity is sold (purchased) on net as an output (input). To characterize uncertain transactions, let there be S mutually exclusive states of nature and denote the netput vector in state s as $\mathbf{y}^s \in \mathbb{R}^L$ where $y_l^s > (<) 0$ implies the lth commodity is sold (purchased) on net as an output (input) in state s. Let $\mathbf{y}^u = (\mathbf{y}^1, \dots, \mathbf{y}^S)$ and $\mathbf{y} = (\mathbf{y}^0, \mathbf{y}^u)$ to economize on notation.

Production possibilities are described by $\mathbf{PPS} \subset \mathbb{R}^{L+L\times S}$, which satisfies the assumptions

A.1 $\mathbf{PPS}$ is non-empty.

A.2 $\mathbf{PPS}$ is closed.

A.3 Free Disposal: If $\mathbf{y} \in \mathbf{PPS}$, $\mathbf{y}' \in \mathbf{PPS}$ for all $\mathbf{y}' \leq \mathbf{y}$.

A.4 Convexity: For all $\mathbf{y}, \mathbf{y}' \in \mathbf{PPS}$ and $\alpha \in [0, 1]$, $\tau\mathbf{y} + (1-\tau)\,\mathbf{y}' \in \mathbf{PPS}$.

A.5 Representable by a continuous and differentiable transformation function $T(\mathbf{y}) \in \mathbb{R}$ where $T(\mathbf{y}) = 0$ implies that $\mathbf{y} \in \mathbf{PPS}$ and $\mathbf{y}' \notin \mathbf{PPS}$ for all $\mathbf{y}' \in \mathbb{R}^{L+L\times S}$, $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$.

Assumptions A.1 and A.2 are the standard guarantees that there is something for a producer to do and the **PPS** contains its boundary. Assumption A.3 is again standard and implies we can increase inputs and still produce the same output or produce less output with the same inputs. Assumption A.4 is also typical implying averages of what is feasible are also feasible. Assumption A.5 is employed mostly to facilitate more widely accessible calculus based arguments.

While this production environment mirrors Chambers and Quiggin (2000), there are differences. Chambers and Quiggin's assume all immediate commodities are inputs, while all uncertain commodities are outputs. We instead focus this distinction exclusively on whether commodity transactions are immediate or uncertain regardless of whether they are used as inputs or outputs, which is consistent with Luenberger's (1995) treatment of production under uncertainty. We choose to permit immediate outputs because they are a common feature of agricultural production, which is particularly well suited for this model given the inherent weather, pest, disease, occupational health, and marketing uncertainties faced by privately held farm businesses. For example, a farmer may choose to rent some land to a neighbor rather than work it, provide custom planting or fertilizer services, or sell stored grain to generate operating capital. Similarly, uncertain inputs are also common. Inclement weather may preclude a pre-emergent herbicide application or spark an unanticipated disease outbreak, leading a farmer to return to his agricultural retailer to trade for or purchase additional inputs such as a post emergent herbicide or fungicide.

Assuming competitive pricing, Chambers and Quiggin (2000) derive a revenue cost function as a dual representation of the technology. This dual representation is analogous to PCT's cost function and facilitates analysis by aggregating optimal commodity sales into uncertain revenue, reducing the dimensionality of the problem without loss of information. For the problem at hand, let $\mathbf{p}^0 \in \mathbb{R}^L_{++}$ be a vector of competitive prices for certain commodities, $\mathbf{p}^s \in \mathbb{R}^L_{++}$ be a vector of competitive prices in state *s* for uncertain commodities, $\mathbf{p}^u = (\mathbf{p}^1, \dots, \mathbf{p}^S)$, and $\mathbf{p} = (\mathbf{p}^0, \mathbf{p}^u)$ for notational convenience.[1] Varying prices across states provides the opportunity to explore market as well as technological uncertainty. In essence, the producer knows that whatever state emerges can influence its prices as well as its feasible input and output combinations. For example, a drought can reduce a farmer's corn yields, while also driving up corn prices depending on how widespread it is.

[1] While one could certainly argue about the prospect of having complete, competitive markets across the spectrum of different states, it is the exogeneity of prices, possibly shadow prices, perceived by the decision maker in various states that matters.

With these prices, consider the problem of maximizing immediate profit, $\pi_0 = \mathbf{p}^0 \cdot \mathbf{y}^0$, while meeting uncertain profit targets, π_s for $s = 1,\ldots, S$:

(1) $\max_{\boldsymbol{y}} \mathbf{p}^0 \cdot \mathbf{y}^0$ subject to $T(\mathbf{y}) = 0$ and $\mathbf{p}^s \cdot \mathbf{y}^s \geq \pi_s^u$ for $s = 1,\ldots, S$.

The solution to this problem are the conditional supply $\mathbf{y}(\mathbf{p}, \boldsymbol{\pi}^u)$, and Lagrange multipliers $\gamma_s(\mathbf{p}, \boldsymbol{\pi}^u)$ for $s = 1,\ldots, S$ and $\delta(\mathbf{p}, \boldsymbol{\pi}^u)$ where $\boldsymbol{\pi}^u$ represents a vector of minimally attainable uncertain profits. Define the immediate profit (IP) function as $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u) = \mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p}, \boldsymbol{\pi}^u)$, which is a natural extension of Chambers and Quiggin's revenue cost function and PCT's cost function. The solution to this problem and the IP function can be shown to satisfy familiar properties.

Proposition 1:[2]

P.1 $\mathbf{y}(\mathbf{p}, \boldsymbol{\pi}^u)$ and $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ are homogeneous of degree zero and one in $\mathbf{p}^0$, and both are homogeneous of degree zero in $\mathbf{p}^s$ and π_s^u for $s = 1,\ldots, S$;

P.2 $(\mathbf{p}^{0\prime} - \mathbf{p}^0) \cdot \left(\mathbf{y}^0\big((\mathbf{p}^{0\prime}, \mathbf{p}^u), \boldsymbol{\pi}^u\big) - \mathbf{y}^0\big((\mathbf{p}^0, \mathbf{p}^u), \boldsymbol{\pi}^u\big)\right) \geq 0$ for all $\mathbf{p}^{0\prime}, \mathbf{p}^0 \in \mathbb{R}^L_{++}$and $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ is convex in $\mathbf{p}^0$; and

P.3 $\frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s} \leq 0$, $y_l^s(\mathbf{p}', \boldsymbol{\pi}^{u\prime}) = -\frac{\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial p_l^s}}{\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s^u}}$, and $y_l^0(\mathbf{p}', \boldsymbol{\pi}^u) = \frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^u)}{\partial p_l^0}$ for $l = 1,\ldots,L$ and $s = 1,\ldots, S$ if $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ is differentiable at $\boldsymbol{\pi}^{u\prime}$ and $\mathbf{p}'$, and $\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s^u} < 0$.

Property P.1 implies only relative prices and uncertain profit targets matter. Property P.2 implies that own price effects for immediate inputs and outputs are non-positive and non-negative (e.g., the law of supply and demand hold for immediate inputs and outputs). Property P.3 implies that increasing uncertain profit in state *s* cannot be accomplished while also increasing certain immediate profit. It also says commodity supplies can be recovered from the IP function provided it is differentiable, which is analogous to Roy's Identity and Shephard's Lemma.

3 Preferences and Optimal Production

[2] The proof of Proposition 1 closely mirrors the proofs of the properties of expenditure, cost, indirect utility, profit, and revenue cost functions, so they are not repeated here. They are however available in the supplementary online appendix.

To determine the optimal production vector and uncertain profits, we characterize preferences using classical utility theory where the goods are a vector of immediate and uncertain profit: $\boldsymbol{\pi} = (\pi_0, \boldsymbol{\pi}^u) \in \mathbb{R}^{S+1}$.[3] Specifically, preferences over $\boldsymbol{\pi}$ are:

A.6 Complete, transitive, and continuous so that they can be represented by a continuous, real valued utility function $W(\boldsymbol{\pi})$.

A.7 Monotonic such that $W(\boldsymbol{\pi}) > W(\boldsymbol{\pi}')$ if $\boldsymbol{\pi} \geq \boldsymbol{\pi}'$ and $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$.

A.8 Twice differentiable where $W_s(\boldsymbol{\pi}) = \frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}$ and $W_{st}(\boldsymbol{\pi}) = \frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_s \partial \pi_t}$ for all *s* and *t*.

We dispense with common convexity assumptions that imply globally risk averse preferences. We also do not use the assumption of additivity between immediate and uncertain profits, which provides latitude to use the model to characterize and explore preferences for the immediate over the delayed as well as the certain over uncertain. Allowing immediate profit to be imperfect substitutes for uncertain profit further differentiates our effort from Debreu (1959) and Luenberger (1995), which assume perfect substitutability over time and states, and Chambers and Quiggin (2000), which assumes perfect substitutability over time. Finally, it is worth noting that a utility function of such generality can represent a fairly wide range of behavioral regularities not captured by risk averse, expected utility (e.g., preference for immediacy and certainty, probability weighting, and risk seeking over losses and risk aversion over gains).

The producer is assumed to use this utility function to choose the optimal feasible production vector:

A.9 $\mathbf{y}(\mathbf{p}) = \{\mathbf{y} \in \mathbf{PPS} | W(\mathbf{p}^0 \cdot \mathbf{y}^0, \ldots, \mathbf{p}^S \cdot \mathbf{y}^S) \geq W(\mathbf{p}^0 \cdot \mathbf{y}^{0\prime}, \ldots, \mathbf{p}^S \cdot \mathbf{y}^{S\prime}) \text{ for } \mathbf{y}' \in \mathbf{PPS}\}$,

which for expositional expedience, we assume

A.10 $\mathbf{y}(\mathbf{p})$ is unique.

This solution can also be obtained using the dual technology representation:

(2a) $$\max_{\boldsymbol{\pi}^u} W(\pi_0(\mathbf{p}, \boldsymbol{\pi}^u), \boldsymbol{\pi}^u),$$

which has the first order condition

(2b) $$W_s(\boldsymbol{\pi}^*) + W_0(\boldsymbol{\pi}^*) \frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^{u*})}{\partial \pi_s} = 0 \text{ or } \theta_s^0(\boldsymbol{\pi}) + \frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^{u*})}{\partial \pi_s} = 0 \text{ for } s = 1, \ldots, S$$

[3] This problem could be attacked in greater detail by assuming profits are explicitly used for consumption similar to Bellemare, Barrett and Just (2013). We instead subsume any such consumption decisions into the utility function in order to maintain a more focused analysis on production decisions.

where $\theta_s^0(\boldsymbol{\pi}) = \frac{W_s(\boldsymbol{\pi}^*)}{W_0(\boldsymbol{\pi}^*)}$ reflects the marginal rate of substitution between immediate and uncertain profit in state *s*. The solution to this problem is a vector of optimal profits depending on prices: $\boldsymbol{\pi}(\mathbf{p}) = \left(\pi_0\left(\mathbf{p}, \boldsymbol{\pi}^u(\mathbf{p})\right), \boldsymbol{\pi}^u(\mathbf{p})\right)$. The uncertain component of this vector can be used with the conditional supply found with equation (1) to determine optimal supply: $\mathbf{y}(\mathbf{p}) = \mathbf{y}\left(\mathbf{p}, \boldsymbol{\pi}^u(\mathbf{p})\right)$.[4]

The solution to this second problem is conveniently illustrated with two states of nature when immediate and uncertain profit are additive in the utility function (i.e., $W(\boldsymbol{\pi}) = W(\pi_1 + \pi_0, \pi_2 + \pi_0)$). Figure 1 provides such an illustration. The IP function embodies the producer's profit possibility frontier (PPF), which is denoted by Π^a in Figure 1. Along this PPF it is impossible to increase profit in any given state without decreasing profit in an alternative state, so it effectively captures the producer's budget constraint. The optimal combination of profit is found at point *a* where the indifference curve W^a is just tangent to Π^a. At this point of tangency, the marginal rate of transformation equals the marginal rate of substitution: $\frac{\frac{\partial \pi_0\left(\mathbf{p}, \boldsymbol{\pi}^{u^*}\right)}{\partial \pi_1}}{\frac{\partial \pi_0\left(\mathbf{p}, \boldsymbol{\pi}^{u^*}\right)}{\partial \pi_2}} = \frac{W_1(\boldsymbol{\pi}^*)}{W_2(\boldsymbol{\pi}^*)}$. Diagrammatically, Figure 1 is much like an intermediate level treatment of consumer demand except for the nonlinear budget constraint.

4 Generalized Arrow-Pratt Measures of Uncertainty

Figure 1 emphasizes that the optimal profit vector is found at a point of tangency between the PPF and the highest possible indifference curve given the PPF. Before exploring the effect of a price change on a producer's optimal profits and supply, the development of a systematic characterization of this point of tangency is worth some consideration. We build insight by redeveloping and expanding Nau's (2003, 2011) generalization of Arrow-Pratt risk aversion coefficients. This generalization can be accomplished by considering how the marginal rate of substitution, $\theta_t^s(\boldsymbol{\pi}) = \left|\frac{d\pi_s}{d\pi_t}\right| = \frac{W_t(\boldsymbol{\pi})}{W_s(\boldsymbol{\pi})}$ for *s*, *t* = 0,…, *S*, varies as profits increase either additively or proportionally. Figure 2(a) illustrates for an equal additive increase, while Figure 2(b) illustrates for a proportional increase assuming we have only two states of nature and additivity between immediate and uncertain profit. Of interest is how the marginal rate of substitution varies as we move from point *a* to *b* in each figure where the marginal rate of substitution equals the slope of the line tangent at each of these points.

[4] See Lemma 2 in the supplementary online appendix.

To measure these changes in the marginal rates of substitution, we first define the directional marginal substitution ($\boldsymbol{\delta}$-MS) coefficients where $\boldsymbol{\delta}$ is a directional vector (e.g., from point *a* to *b*):

Definition: The directional marginal substitution ($\boldsymbol{\delta}$-MS) coefficients are

$\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta}) = -\frac{\sum_{r=0}^{S} \delta_r W_{sr}(\boldsymbol{\pi})}{W_s(\boldsymbol{\pi})}$ for $s = 0,\ldots,S$ and $\boldsymbol{\delta} \in \mathbb{R}^{S+1}$.

The $\boldsymbol{\delta}$-MS coefficients are a generalization of Arrow-Pratt risk aversion coefficients. To verify this, suppose $W(\boldsymbol{\pi}) = \sum_{s=1}^{S} \phi_s\, u(\pi_s + \pi_0)$ where $\phi_s \geq 0$ is the probability of state *s* such that $\sum_{s=1}^{S} \phi_s = 1$ and $u(\cdot)$ is a Bernoulli utility function. Differentiation and substitution then imply $\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta}) = -(\delta_s + \delta_0)\frac{u''(\pi_s+\pi_0)}{u'(\pi_s+\pi_0)}$ for $s = 1,\ldots,S$. Therefore, for $\mathbf{1}^{S+1}$ equal to a vector of $S + 1$ ones, $\rho_s\left(\boldsymbol{\pi}, \frac{1}{2} \times \mathbf{1}^{S+1}\right)$ is the Arrow-Pratt coefficient of absolute risk aversion, while $\rho_s(\boldsymbol{\pi}, \boldsymbol{\pi})$ is the Arrow-Pratt coefficient of relative risk aversion.

More generally, consider how $\theta_t^s(\boldsymbol{\pi} + \boldsymbol{\delta}\varepsilon)$ varies with ε at $\varepsilon = 0$:

$$\text{(3)} \qquad \left.\frac{d\theta_t^s(\boldsymbol{\pi}+\boldsymbol{\delta}\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \theta_t^s(\boldsymbol{\pi})\big(\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta}) - \rho_t(\boldsymbol{\pi}, \boldsymbol{\delta})\big).$$

Equation (3) shows that the change in the marginal rates of substitution as profits change is proportional to the difference in the $\boldsymbol{\delta}$-MS coefficients. Therefore, the Arrow-Pratt coefficients of risk aversion systematically measure how the marginal rates of substitution change as profit or income in each state change either additively or proportionally, a concept that is broadly applicable beyond uncertain production or production in general. Thus, we refer to $\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta})$ for $s = 0,\ldots, S$ as the $\boldsymbol{\delta}$-MS coefficients because they embody how the marginal rates of substitution change as the consumption of some vector of goods $\boldsymbol{\pi}$ changes in the direction of $\boldsymbol{\delta}$ where $\boldsymbol{\pi}$ could easily be thought of as a vector of household consumption goods instead of state-contingent profit.

The $\boldsymbol{\delta}$-MS coefficients differ from Nau's development in three ways. First, Nau only considers state contingent income — any immediate income is additive to uncertain income. Second, Nau uses an $S\times S$ matrix of generalized Arrow-Pratt coefficients to characterize uncertainty preferences, while as with Arrow-Pratt, our $\boldsymbol{\delta}$-MS coefficients are differentiated only by the alternative states, including the certain state. Third, Nau's characterization is analogous to the Arrow-Pratt coefficient of absolute risk aversion, while ours is flexible enough to encompass notions of relative as well as absolute and many other potentially useful measures of risk or uncertain preferences.

Luenberger (1995, p. 394) also offers a generalization of Arrow-Pratt risk aversion in the context of state contingent preferences. This generalization differs from ours because it measures changes in the curvature of indifference curves about the vector of equal state contingent incomes/profits (i.e., about points of certainty), while ours measures changes in curvature about arbitrary vectors of immediate and state contingent incomes/profits.

With the $\boldsymbol{\delta}$-MS coefficients defined, we can further define the $S \times S$ uncertainty aversion matrix analogous to Nau's (2003) risk aversion matrix:

$$\text{(4)} \qquad \mathbf{P} = \begin{bmatrix} \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_1}\right) - \rho_1\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_1}\right) & \cdots & \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_S}\right) - \rho_1\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_S}\right) \\ \vdots & \ddots & \vdots \\ \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_1}\right) - \rho_S\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_1}\right) & \cdots & \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_S}\right) - \rho_S\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_S}\right) \end{bmatrix}.$$

Intuitively, this matrix reflects uncertainty aversion because it measures how the marginal rates of substitution between immediate and uncertain profit change as uncertain profits increase. We also define the $S \times 1$ matrix that we refer to as the patience aversion matrix because it measures how marginal rates of substitution between uncertain and immediate profits change as immediate profit increases:

$$\text{(5)} \qquad \boldsymbol{\Delta} = \begin{bmatrix} \rho_1\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) - \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) \\ \vdots \\ \rho_S\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) - \rho_0\left(\boldsymbol{\pi}^*, \frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) \end{bmatrix}.$$

The $\boldsymbol{\delta}$-MS coefficients characterize how uncertainty and temporal preferences are changing as profits change. This concept extends to characterizing how the marginal rate of transformation is changing as uncertain profits or prices change:

Definition: The profit-directional marginal transformation ($\boldsymbol{\delta}^{\boldsymbol{\pi}}$-MT) coefficients are

$$\tau_s^{\boldsymbol{\pi}}(\mathbf{p}, \boldsymbol{\pi}^u, \boldsymbol{\delta}^{\boldsymbol{\pi}}) = \frac{\sum_{r=1}^{S} \delta_r^{\pi} \frac{\partial^2 \pi_0(\mathbf{p}, \boldsymbol{\pi}^u)}{\partial \pi_s \partial \pi_r}}{\frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^u)}{\partial \pi_s}} \text{ for } s = 1,\ldots,S \text{ and } \boldsymbol{\delta}^{\boldsymbol{\pi}} \in \mathbb{R}^S.$$

Definition: The price-directional marginal transformation ($\boldsymbol{\delta}^{\mathbf{p}}$-MT) coefficients are

$$\tau_s^{\mathbf{p}}(\mathbf{p}, \boldsymbol{\pi}^u, \boldsymbol{\delta}^{\mathbf{p}}) = \frac{\sum_{r=0}^{S} \sum_{l=1}^{L} \delta_r^{lp} \frac{\partial^2 \pi_0(\mathbf{p}, \boldsymbol{\pi}^u)}{\partial \pi_s \partial p_r^l}}{\frac{\partial \pi_0\left(\mathbf{p}, \boldsymbol{\pi}^{u*}\right)}{\partial \pi_s}} \text{ for } s = 1,\ldots,S \text{ and } \boldsymbol{\delta}^{\mathbf{p}} \in \mathbb{R}^{(S+1)\times L}.$$

The marginal rate of transformation is defined as $\varphi_t^s(\boldsymbol{\pi}^u) = \left|\frac{d\pi_s}{d\pi_t}\right| = \frac{\frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^u)}{\partial \pi_t}}{\frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^u)}{\partial \pi_s}}$ for s, t = 1,…, S. How it varies as uncertain profits change in the direction of $\boldsymbol{\delta}^{\boldsymbol{\pi}} \in \mathbb{R}^S$ is

$$\text{(6)} \qquad \left.\frac{d\varphi_t^s(\mathbf{p}, \boldsymbol{\pi}^u + \boldsymbol{\delta}^{\boldsymbol{\pi}} \varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \varphi_t^s(\boldsymbol{\pi}^u)\left(\tau_t^{\boldsymbol{\pi}}(\mathbf{p}, \boldsymbol{\pi}^u, \boldsymbol{\delta}^{\boldsymbol{\pi}}) - \tau_s^{\boldsymbol{\pi}}(\mathbf{p}, \boldsymbol{\pi}^u, \boldsymbol{\delta}^{\boldsymbol{\pi}})\right).$$

Similarly, how it varies as prices change in the direction of $\boldsymbol{\delta}^{\mathbf{p}} \in \mathbb{R}^{(S+1)\times L}$ is

(7) $$\left.\frac{d\varphi_t^s(\mathbf{p}+\boldsymbol{\delta}^{\mathbf{p}}\varepsilon,\boldsymbol{\pi}^u)}{d\varepsilon}\right|_{\varepsilon=0} = \varphi_t^s(\boldsymbol{\pi}^u)\left(\tau_t^{\mathbf{p}}(\mathbf{p},\boldsymbol{\pi}^u,\boldsymbol{\delta}^{\mathbf{p}}) - \tau_s^{\mathbf{p}}(\mathbf{p},\boldsymbol{\pi}^u,\boldsymbol{\delta}^{\mathbf{p}})\right).$$

With these definitions and equations (6) and (7), we can define technological and market price uncertainty matrices that are the technological and market analogous to Nau's uncertain preference characterization, and our uncertain and temporal preference characterizations:

(8) $$\mathbf{T}_{\boldsymbol{\pi}^u} = \begin{bmatrix} \tau_1^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \tau_1^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right) \\ \vdots & \ddots & \vdots \\ \tau_S^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \tau_S^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right) \end{bmatrix} \text{ and } \mathbf{T}_{p_k^t} = \begin{bmatrix} \tau_1^{\mathbf{p}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\mathbf{p}}{\partial p_k^t}\right) \\ \vdots \\ \tau_S^{\mathbf{p}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\mathbf{p}}{\partial p_k^t}\right) \end{bmatrix}.$$

A similar characterization of market uncertainty in the context of risky household production with expected utility preferences is found in Bellemare, Barrett and Just (2013), while we are unaware of similar characterizations of technological uncertainty. As will soon be evident, these systematic characterizations of preferences, technology, and markets are key ingredients in the optimal response to changes in competitive prices as well as other exogenous factors.

5 Slutsky Price Analysis

The Slutsky equation from consumer theory uses Hicksian demand and the expenditure function to decompose the effect of a price change on Marshallian demand into an income and substitution effect. A similar strategy using the IP function and totally differentiating equation (2b) yields:

Proposition 2: The effect of a change in the price of commodity $k = 1,\ldots, L$ in state $t = 0,\ldots, S$ on the commodity supply $\mathbf{y}(\mathbf{p})$ is

$$\frac{\partial\mathbf{y}(\mathbf{p})}{\partial p_k^t} = \frac{\partial\mathbf{y}\left(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})\right)}{\partial p_k^t} + \frac{\partial\mathbf{y}\left(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})\right)}{\partial\boldsymbol{\pi}^u}\frac{\partial\boldsymbol{\pi}^u(\mathbf{p})}{\partial p_k^t}$$

where

$$\frac{\partial\boldsymbol{\pi}^u(\mathbf{p})}{\partial p_k^t} = \left(\mathbf{P}+\boldsymbol{\Delta}\boldsymbol{\theta}^0-\mathbf{T}_{\boldsymbol{\pi}^u}\right)^{-\mathbf{1}}\left(\boldsymbol{\Delta}\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial p_k^t}+\mathbf{T}_{p_k^t}\right) \text{ and}$$

$$\boldsymbol{\theta}^0 = \left(\theta_1^0\left(\boldsymbol{\pi}(\mathbf{p})\right) \quad \cdots \quad \theta_S^0\left(\boldsymbol{\pi}(\mathbf{p})\right)\right).$$

The first equation in Proposition 2 breaks the effect of the price change on the commodity supplies into a substitution effect $\left(\frac{\partial\mathbf{y}(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))}{\partial p_k^t}\right)$ and an income effect $\left(\frac{\partial\mathbf{y}(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))}{\partial\boldsymbol{\pi}^u}\frac{\partial\boldsymbol{\pi}^u(\mathbf{p})}{\partial p_k^t}\right)$. From property P.2, we know own-price substitution effects are non-negative (non-positive) if the commodity is a certain output (input). Otherwise, they are ambiguous. Given unambiguous own-price effects for certain inputs and outputs, it is intuitive to categorize cross-price effects in terms of substitute versus complement commodities: $\frac{\partial |y_l^s(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))|}{\partial p_k^0} > 0$ versus $\frac{\partial |y_l^s(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))|}{\partial p_k^0} < 0$ for input $y_l^0(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})) < 0$ where $l \neq k$, or $\frac{\partial |y_l^s(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))|}{\partial p_k^0} < 0$ versus $\frac{\partial |y_l^s(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))|}{\partial p_k^0} > 0$ for output $y_l^0(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})) > 0$ where $l \neq k$. Both own- and cross-price substitution effects are ambiguous for uncertain inputs and outputs, which makes categorization based on the notion of substitute versus complement commodities more challenging. This challenge is not exclusive to uncertain choice however with similar challenges emerging in certain production. These challenges have been met by appealing to supermodularity (see Topkis 1998), which is shown to extend to uncertain production by Chambers and Quiggin (2008).

The income effect reflects how the optimal use of the commodity changes due to a change in optimal uncertain profits. The first part of this term $\left(\frac{\partial \mathbf{y}(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))}{\partial \boldsymbol{\pi}^u}\right)$ is generally ambiguous depending on whether an input or output is in demand for producing a particular uncertain profit. The common taxonomy for such effects is normal versus inferior commodities. If a commodity is normal (inferior) for a particular state's uncertain profit, then its purchase or sale will be increasing (decreasing) in uncertain profit in that state: $\frac{\partial |y_l^s(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p}))|}{\partial \pi_v} > (<) 0$. The second part of this term and the second equation in Proposition 2 reflects how the price change affects the optimal supply of uncertain profit in a given state. This term depends crucially on our uncertainty and patience aversion matrices, and technological and market uncertainty matrices.

The net effect of the price change on the demand for uncertain profit can be conveniently illustrated in terms of substitution and income effects when there are two states of nature and additive immediate and uncertain profit in the utility function. Figure 3 provides such an illustration for a price change that reduces profitability in both states such as an increase in land rental rates for a farmer. The optimal combination of profit before the price change is at point *a* where the indifference curve W^a is just tangent to the PPF Π^a. Now suppose an increase in land rental rates drives the PPF down to Π^b resulting in a new optimum at point *b* where the indifference curve W^b is just tangent to the PPF Π_0^b. At this new optimum, profit in state 2 is relatively less costly to produce, which is seen by the comparison of the slope of the tangent through point *b* to the slope of the tangent through point *a*. Overall however, production becomes more costly leading to a scaling back and lower welfare as reflected by the move to a lower indifference curve. To discern the change in profits attributable to the substitution and income effect, we can shift the new line of tangency at point *b* parallel until it is just tangent to the original PPF at point *c*. The difference in total profits between points a and *c* ($\pi_1^c - \pi_1^a$ and $\pi_2^c - \pi_2^a$) reflects a substitution effect — a movement along the original PPF. This substitution effect is driven by how the producer's tolerance for uncertainty changes as it scales back its production. In this example, the marginal rate of substitution increases implying the producer is less willing to gives up profit in state 1 for profit in state 2. Since state 2 profit exceeds state 1 profit at point *b*, this suggests the producer has become less tolerant of uncertainty. The difference in profits between points *c* and *b* ($\pi_1^b - \pi_1^c$ and $\pi_2^b - \pi_2^c$) reflect the income effect — a jump from the original to the new PPF. The income effect is driven by how much the producer's welfare changes due to the increase in land rental rates. This illustration is recognizable as an intermediate level graphical treatment of the Slutsky income and substitution effects except for the nonlinear budget constraint.

6 Sandmo Revisited

We further explore the utility of our $\boldsymbol{\delta}$-MS coefficients and demonstrate how they are a natural generalization of Arrow-Pratt risk aversion coefficients by revisiting Sandmo's (1971) comparative static analysis of a producer's choice of output when facing price risk. In this analysis, Sandmo assumed a producer chooses a certain output x given the cost function $F(x) = C(x) + B$ where $C(x) \geq 0$ are variable costs such that $C(0) = 0$ and $C'(x) > 0$, and $B \geq 0$ are fixed costs. Price risk was characterized using a continuous distribution function, which we translate to the state contingent perspective by assuming $p^s + \Delta > 0$ is the price the producer receives for output in state *s* and $\Delta \geq 0$ is an additive shift parameter to facilitate comparative static analysis. Without loss of generality, we assume $p^{s+1} > p^s$ for $s = 1,\ldots, S - 1$. Preferences were characterized using risk averse expected utility, so immediate costs and uncertain revenues were treated additively (e.g., $W(\boldsymbol{\pi}) = \sum_{s=1}^{S} \phi_s \, u(\pi_s + \pi_0)$ where $u'(\cdot) > 0$ and $u''(\cdot) < 0$). We start with our more general utility function and explore what additional restrictions, if any, are sufficient to recover Sandmo's results. Given the profit tax rate $\tau \in (0, 1)$, the producer's problem is

(9a) $$\max_{x\geq 0} W(\boldsymbol{\pi}) \text{ where } \boldsymbol{\pi} = (-F(x), (p^1 + \Delta)x, \ldots, (p^S + \Delta)x)(1 - \tau),$$

which has the first-order condition

(9b) $$D = \sum_{s=1}^{S} W_s(\boldsymbol{\pi}^*)(p^s + \Delta) - W_0(\boldsymbol{\pi}^*)C'(x^*) = 0$$

where * denote optimal values. The solution to this problem, assuming it exists, depends on prices, fixed costs, the additive shift parameter, and tax rate: $x(\mathbf{p}, B, \Delta, \tau)$.

The key comparative static results derived by Sandmo include:

(a) Positive/constant/negative relationship between optimal output and fixed costs (B) when preferences exhibit increasing/constant/decreasing absolute Arrow-Pratt risk aversion.

(b) Positive relationship between optimal output and an additive increase in output prices (Δ) when preferences exhibit decreasing or constant Arrow-Pratt risk aversion.

(c) Positive/constant/negative relationship between optimal output and the tax rate (τ) when preferences exhibit increasing/constant/decreasing relative Arrow-Pratt risk aversion.

To explore the robustness of these results, given our state contingent perspective and generalization of Arrow-Pratt's risk aversion coefficients, we first propose two new definitions:

Definition: Preferences exhibit increasing/constant/decreasing **δ**-uncertainty aversion (**δ**-UA) at **π** if $\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta}) >/=/< \rho_t(\boldsymbol{\pi}, \boldsymbol{\delta})$ when $\pi_s > \pi_t$ for all s, t = 1,…,S.

Definition: Preferences exhibit positive/neutral/negative **δ**-patience aversion (**δ**-PA) at **π** if $\rho_s(\boldsymbol{\pi}, \boldsymbol{\delta}) >/=/< \rho_0(\boldsymbol{\pi}, \boldsymbol{\delta})$ for all s = 1,…,S.

Since $\rho_s\left(\boldsymbol{\pi}, \frac{1}{2} \times \mathbf{1}^{S+1}\right)$ and $\rho_s(\boldsymbol{\pi}, \boldsymbol{\pi})$ equal the absolute and relative Arrow-Pratt risk aversion coefficients in the state contingent model with expected utility preferences, increasing/ constant/ decreasing absolute and relative Arrow-Pratt risk aversion are special cases of increasing/ constant/ decreasing **δ**-UA. Alternatively, positive/ neutral/ negative **δ**-PA are novel, but related to the notion of risk averse, neutral and seeking preferences. It is also important to note that these definitions are local as they depend on a specific **π**.

The implicit function theorem implies $\frac{\partial x(\mathbf{p}, B, \Delta, \tau)}{\partial \varepsilon} = -\frac{\frac{\partial D}{\partial \varepsilon}}{\frac{\partial D}{\partial x}}$ for $\varepsilon \in \{B, \Delta, \tau\}$. For a unique maximum $\frac{\partial D}{\partial x} < 0$, so how optimal output changes depends on the sign of $\frac{\partial D}{\partial \varepsilon}$, which for $\varepsilon = B$, Δ, and τ, equals

(10a) $$\frac{\partial D}{\partial B} = \sum_{s=1}^{S} \rho_s\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial B}\right) W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*) C\prime(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right)$$
$$+ W_0(\boldsymbol{\pi}^*) C'(x^*) \sum_{s=1}^{S} \frac{W_s(\boldsymbol{\pi}^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\left(\rho_s\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial B}\right) - \rho_0\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial B}\right)\right),$$

(10b) $$\frac{\partial D}{\partial \Delta} = -\sum_{s=1}^{S} \rho_s\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial \Delta}\right) W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*) C'(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right)$$
$$- W_0(\boldsymbol{\pi}^*) C'(x^*) \sum_{s=1}^{S} \frac{W_s(\boldsymbol{\pi}^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\left(\rho_s\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial \Delta}\right) - \rho_0\left(\boldsymbol{\pi}, \frac{\partial \boldsymbol{\pi}}{\partial \Delta}\right)\right)$$
$$+ \sum_{s=1}^{S} W_s(\boldsymbol{\pi}^*),$$ and

(10c) $$\frac{\partial D}{\partial \tau} = \sum_{s=1}^{S} \rho_s\left(\boldsymbol{\pi},\frac{\partial \boldsymbol{\pi}}{\partial \tau}\right) W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)C'(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right)$$

$$+W_0(\boldsymbol{\pi}^*)C'(x^*)\sum_{s=1}^{S}\frac{W_s(\boldsymbol{\pi}^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\left(\rho_s\left(\boldsymbol{\pi},\frac{\partial \boldsymbol{\pi}}{\partial \tau}\right) - \rho_0\left(\boldsymbol{\pi},\frac{\partial \boldsymbol{\pi}}{\partial \tau}\right)\right).$$

The first terms on the right-hand-side of equations (10a) – (10c) capture how a producer's tolerance for uncertainty drives its response to the changing production environment. Analogous expressions found in Sandmo's analysis are evaluated by appealing to increasing/ constant/ decreasing absolute and relative Arrow-Pratt risk aversion. To show how these arguments are just as applicable in our more general framework, we focus on equation (10a) assuming preferences satisfy increasing $\frac{\partial \boldsymbol{\pi}}{\partial B}$-UA. Define $\bar{p} = \frac{W_0(\boldsymbol{\pi}^*)C'(q^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)} - \Delta$ and $\bar{\rho} = \min\left\{\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial B}\right) | s = 1,\ldots,S \text{ and } p^s \geq \bar{p}\right\}$. For $p^s \geq (<)\bar{p}$,

(11) $$\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial B}\right) \geq (<)\bar{\rho} \text{ and } W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)c\prime(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right) \geq (<)0.$$

Multiplication then implies:

(12) $$\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial B}\right) W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)c\prime(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right) \geq \bar{\rho} W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)c'(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right)$$

with strict inequality for some *s*. Summing equation (12) for *s* = 1,…, *S* yields the desired result

(13) $$\sum_{s=1}^{S} \rho_s\left(\boldsymbol{\pi},\frac{\partial \boldsymbol{\pi}}{\partial B}\right) W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)c\prime(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right) > \bar{\rho}\sum_{s=1}^{S} W_s(\boldsymbol{\pi}^*)\left(p^s + \Delta - \frac{W_0(\boldsymbol{\pi}^*)c'(x^*)}{\sum_{t=1}^{S} W_t(\boldsymbol{\pi}^*)}\right) = 0.$$

Through similar arguments, it is possible to establish that the first term in equations (10a) and (10c) will be positive/ zero/ negative when preferences exhibit increasing/constant/decreasing $\frac{\partial \boldsymbol{\pi}}{\partial B}$ - UA and $\frac{\partial \boldsymbol{\pi}}{\partial \tau}$ - UA. Alternatively, the first term in equation (10b) will be negative/ zero/ positive when preferences exhibit increasing/constant/decreasing $\frac{\partial \boldsymbol{\pi}}{\partial \Delta}$ - UA.

The second term in equations (10a) – (10c) captures how the producer's patience aversion drives its response to the changing economic environment. An analogous effect does not appear in Sandmo's analysis due to the assumption that immediate costs are additive with uncertain revenue. By definition, whether this term is positive/ zero/ negative depends on whether preferences exhibit positive/ neutral/ negative $\frac{\partial \boldsymbol{\pi}}{\partial B}$-PA and $\frac{\partial \boldsymbol{\pi}}{\partial \tau}$-PA for equations (10a) and (10c), or negative/ neutral/ positive $\frac{\partial \boldsymbol{\pi}}{\partial \Delta}$-PA for equation (10b).

The third term in equation (10b) is positive regardless of the producer's tolerance for uncertainty or patience. An analogous effect is found in Sandmo's analysis, which he refers to as a substitution effect because it is optimal to produce more when you can sell for a higher price.

Our final proposition summarizes these findings as sufficiency conditions:

Proposition 3: For a producer facing price uncertainty, there is a

(a) positive/ constant/ negative relationship between optimal output and fixed costs (B) when preferences exhibit increasing/constant/decreasing $\frac{\partial \boldsymbol{\pi}}{\partial B}$-UA, and positive or neutral/ neutral/ negative or neutral $\frac{\partial \boldsymbol{\pi}}{\partial B}$-PA;

(b) positive relationship between optimal output and an additive increase in output prices (Δ) when preferences exhibit constant or decreasing $\frac{\partial \boldsymbol{\pi}}{\partial \Delta}$-UA and negative or neutral $\frac{\partial \boldsymbol{\pi}}{\partial \Delta}$-PA; and

(c) positive/ constant/ negative relationship between optimal output and the tax rate (τ) when preferences exhibit increasing/constant/ decreasing $\frac{\partial \boldsymbol{\pi}}{\partial \tau}$-UA and positive or neutral/ neutral/ negative or neutral $\frac{\partial \boldsymbol{\pi}}{\partial \tau}$-PA.

Proposition 3 reveals that Sandmo's results are fairly robust for uncertain preferences when preferences also exhibit neutral $\boldsymbol{\delta}$-PA, which is the case when uncertain and immediate profit is additive. With non-neutral $\boldsymbol{\delta}$-PA preferences, Sandmo's results may still hold but are no longer guaranteed because of the producer's preference for patience also matters. For example, an increase in fixed costs unequivocally decreases immediate and certain profit relative to future uncertain profits. To the extent a producer prefers immediate profit it will want to respond with a decrease in its output to reduce its immediate variable costs relative to its future revenue. Such a desire will be reinforced and Sandmo's result will hold if the producer's tolerance for uncertainty decreases as its welfare decreases due to higher fixed costs. This occurs because the producer will be inclined to reduce output to also reduce its uncertain revenue relative to its immediate and certain variable costs. Alternatively, Sandmo's result could fail if the producer's tolerance for uncertainty increases with the welfare decrease because the producer will also have a desire to increase output in order to increase uncertain revenue relative to immediate and certain costs. Whether the preference for patience or tolerance for uncertainty prevails then becomes an empirical question.

A final remark about Proposition 3 is that it applies locally and does not require the assumption of risk aversion. Even though Sandmo assumed risk averse preferences, he did not use it in his original comparative static proofs except to say it guaranteed a unique maximum. While risk aversion may be a sufficient condition for a maximum with expected utility preferences, it is not necessary with expected utility or more general preferences.

Therefore, these results will hold even if a producer is not risk averse, as typically defined, provided a unique optimal output is still obtained.

7 Conclusions

Recent efforts to reconnect the theory of uncertain production to the foundations of certain producer and consumer theory (PCT) have made remarkable progress. Yet, these efforts have also remained tethered to concepts that are more appealing in a world of reasonably quantifiable objective risk, than in a world of difficult to quantify subjective risk or uncertainty. In this article, we seek to sever one more restraint to the synthesis of certain and uncertain production that draws on concepts equally applicable to both by characterizing how a change in competitive prices affects optimal inputs and outputs using the concepts of substitution and income effects and their associate taxonomies of substitute versus complement and normal versus inferior commodities. These characterizations do not rely on notions of risk averse preferences, even in a general sense, or probabilistic sophistication. They do rely on generalizations of Arrow-Pratt's absolute and relative risk aversion, which characterize risk seeking and neutral as well as risk averse preferences, and are demonstrably a systematic characterization of the marginal rate of substitution.

The value of our effort is pedagogical and empirical as well as theoretical. Pedagogically, our further development of PCT foundations for uncertain production facilitates instruction by making it possible to convey important concepts of risk and uncertainty with tools that are taught to students beginning with first principles. Empirically, by removing any reference to a producer's subjective probabilities, our framework does not face the difficulties of uniquely identifying risk preferences and perceptions as well as the other major empirical challenges to risky production reviewed by Just et al. (2010). Furthermore, viewed from the PCT perspective, a primary challenge to the estimation of a certain profit function is incomplete information on uncertain prices and net commodity supplies. Yet, similar issues have historically challenged attempts to estimate cost and expenditure systems when information on prices and quantities are incomplete. This challenge has been met by coming up with clever empirical strategies that do not require this missing information. Thus, a fertile direction for future empirical work on uncertain production may well be the exploration of how existing econometric tools can be redeployed to the interpretation of the incomplete data collected on production decisions in environments with substantial risk or uncertainty.

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Figure 1: Illustration of optimal profit determination in an uncertain world.

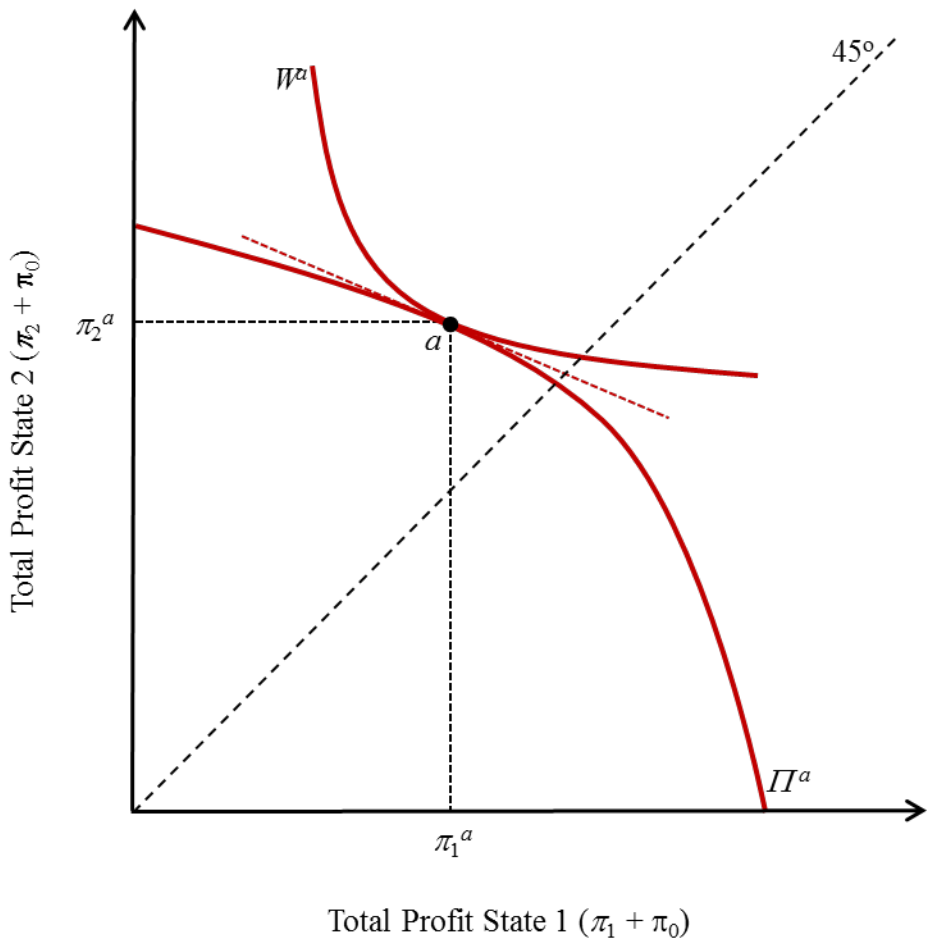


Figure 2: Illustration of changes in the marginal rate of substitution as uncertain profits increase (a) additively and (b) proportionally.

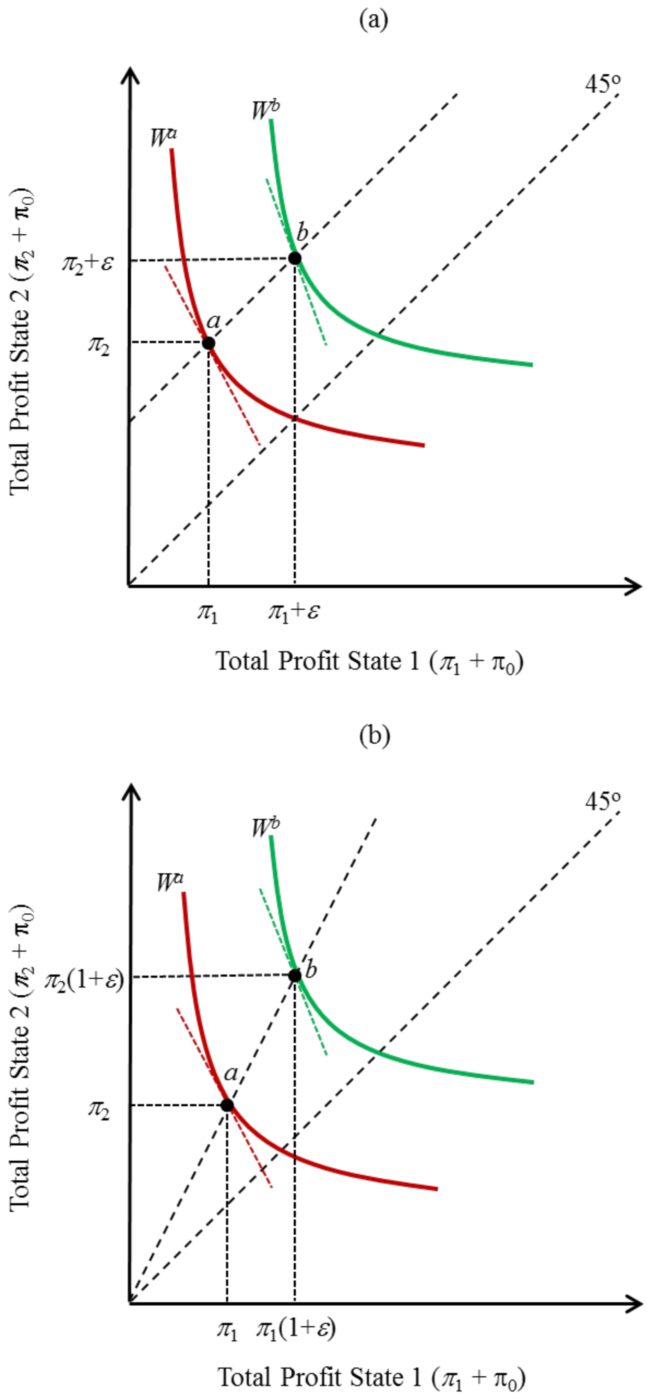
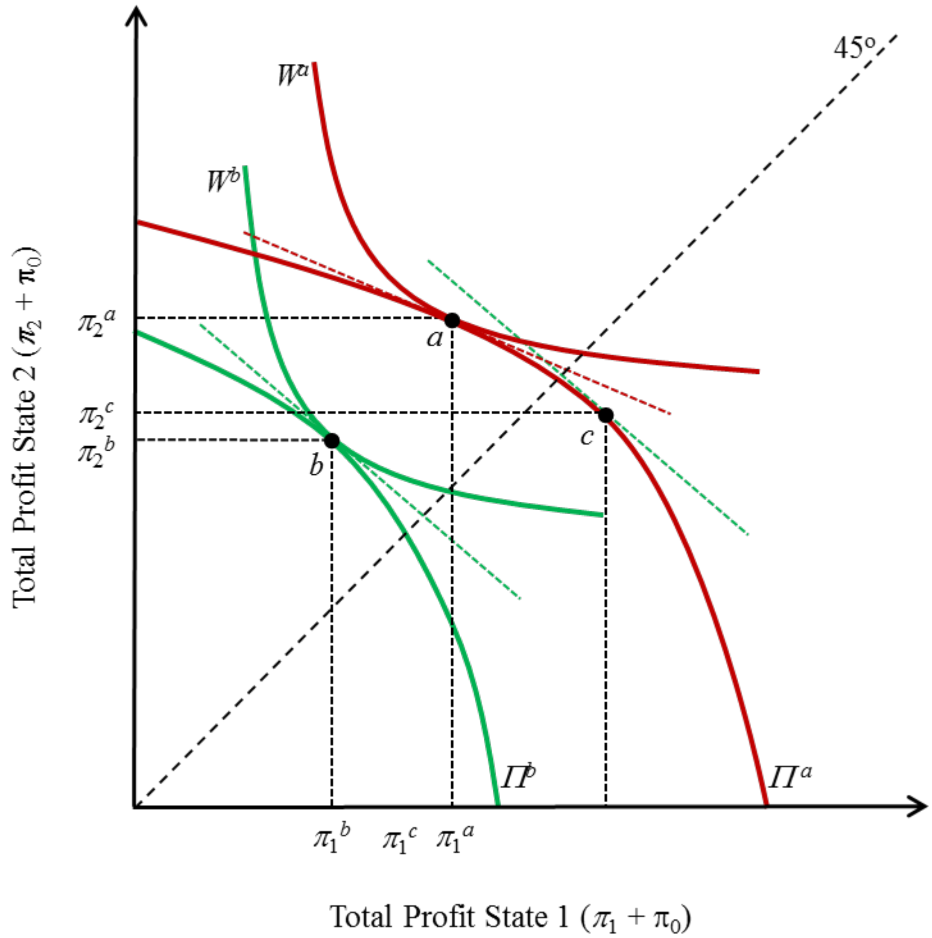


Figure 3: Substitution (point a to c) and income effects (point c to b) with uncertain production.



Online Supplemental Appendix for

"Slutsky, Let Me Introduce You to Arrow-Pratt: Competitive Price Effects with Uncertain Production"

Notation and Assumptions

Certain commodity transactions are denoted by the netput vector $\mathbf{y^0} \in \mathbb{R}^L$ where $y_l^0 > (<) 0$ implies the *l*th commodity is sold (purchased) on net as an output (input). Uncertain commodity transactions are denoted by the netput vector $\mathbf{y}^s \in \mathbb{R}^L$ for states *s* = 1,…, *S* where $y_l^s > (<) 0$ implies the *l*th commodity is sold (purchased) on net as an output (input) in state *s*. Let $\mathbf{y}^u = (\mathbf{y}^1, \dots, \mathbf{y}^S)$ and $\mathbf{y} = (\mathbf{y}^0, \mathbf{y}^u)$ to economize on notation. Production possibilities are described by $\mathbf{PPS} \subset \mathbb{R}^{L(S+1)}$ and assumed to satisfy:

A.1 $\mathbf{PPS}$ is non-empty.

A.2 $\mathbf{PPS}$ is closed.

A.3 Free Disposal: If $\mathbf{y} \in \mathbf{PPS}$, $\mathbf{y}' \in \mathbf{PPS}$ for all $\mathbf{y}' \leq \mathbf{y}$.

A.4 Convexity: For all $\mathbf{y}, \mathbf{y}' \in \mathbf{PPS}$ and $\alpha \in [0,1]$, $\tau\mathbf{y} + (1-\tau)\,\mathbf{y}' \in \mathbf{PPS}$.

A.5 Representable by a continuous and differentiable transformation function $T(\mathbf{y}) \in \mathbb{R}$ where $T(\mathbf{y}) = 0$ implies that $\mathbf{y} \in \mathbf{PPS}$ and $\mathbf{y}' \notin \mathbf{PPS}$ for all $\mathbf{y}' \in \mathbb{R}^{L+L\times S}$, $\mathbf{y}' \geq \mathbf{y}$ and $\mathbf{y}' \neq \mathbf{y}$.

Preferences are characterized over bundles of certain profit $\pi_0 \in \mathbb{R}$ and a vector of state contingent uncertain profit $\boldsymbol{\pi}^u \in \mathbb{R}^S$ such that $\boldsymbol{\pi} = (\pi_0, \boldsymbol{\pi}^u) \in \mathbb{R}^{S+1}$. Assumptions employed for these preferences are

A.6 Complete, transitive, and continuous so that they can be represented by a continuous, real valued utility function $W(\boldsymbol{\pi})$.

A.7 Monotonic such that $W(\boldsymbol{\pi}) > W(\boldsymbol{\pi}')$ if $\boldsymbol{\pi} \geq \boldsymbol{\pi}'$ and $\boldsymbol{\pi} \neq \boldsymbol{\pi}'$.

A.8 Twice differentiable where $W_s(\boldsymbol{\pi}) = \frac{\partial W(\boldsymbol{\pi})}{\partial \pi_s}$ and $W_{st}(\boldsymbol{\pi}) = \frac{\partial^2 W(\boldsymbol{\pi})}{\partial \pi_s \partial \pi_t}$ for all *s* and *t*.

The producer's objective is assumed to be utility maximization with competitive input and output markets. Let certain competitive commodity prices be $\mathbf{p}^0 \in \mathbb{R}^L_{++}$, while uncertain competitive commodity prices are $\mathbf{p}^s \in \mathbb{R}^L_{++}$ for states *s* = 1,…, *S*. For notational convenience, $\mathbf{p}^u = (\mathbf{p}^1, \dots, \mathbf{p}^S)$ and $\mathbf{p} = (\mathbf{p}^0, \mathbf{p}^u)$. Profits are defined as $\pi_s = \mathbf{p}^s \cdot \mathbf{y}^s$ for *s* = 0,…, *S* such that producer's objective is

A.9 $\mathbf{y}(\mathbf{p}) = \{\mathbf{y} \in \mathbf{PPS} | W(\mathbf{p}^0 \cdot \mathbf{y}^0, \dots, \mathbf{p}^S \cdot \mathbf{y}^S) \geq W(\mathbf{p}^0 \cdot \mathbf{y}^{0\prime}, \dots, \mathbf{p}^S \cdot \mathbf{y}^{S\prime}) \text{ for all } \mathbf{y}' \in \mathbf{PPS}\}$.

For expositional convenience, we also assume

A.10 $\mathbf{y}(\mathbf{p})$ is a nonempty, singleton set.

Proofs of Lemmas and Propositions

We first establish that production is efficient, so we can use our transformation function to characterize production possibilities in subsequent analysis.

Lemma 1: Production is efficient — If $\mathbf{y}' \geq \mathbf{y}(\mathbf{p})$ and $\mathbf{y}' \neq \mathbf{y}(\mathbf{p})$**,** then $\mathbf{y}' \notin \mathbf{PPS}$**.**

Proof: Suppose this is not the case such that there exists a $\mathbf{y}' \in \mathbf{PPS}$ such that $\mathbf{y}' \geq \mathbf{y}(\mathbf{p})$ and $\mathbf{y}' \neq \mathbf{y}(\mathbf{p})$**.** If $\mathbf{y}' \geq \mathbf{y}(\mathbf{p})$ and $\mathbf{y}' \neq \mathbf{y}(\mathbf{p})$ then $\pi_s' = \mathbf{p}^s \cdot \mathbf{y}^{s\prime} \geq \mathbf{p}^s \cdot \mathbf{y}^s = \pi_s$ for all *s* and $\pi_s' > \pi_s$ for some *s*. By A.7, $W(\boldsymbol{\pi}') > W(\boldsymbol{\pi})$, which contradicts A.9. Q.E.D.

Proof of our first proposition is facilitated by defining the set $\boldsymbol{\Omega}(\mathbf{p}^u, \boldsymbol{\pi}^u) = \{\mathbf{y} \in \mathbf{PPS} \,|T(\mathbf{y}) = 0 \text{ and } \mathbf{p}^s \cdot \mathbf{y}^s \geq \pi_s^u \text{ for } s = 1, \dots, S\}$, which includes all efficient production vectors that yield a vector of uncertain profit that is at least as large as $\boldsymbol{\pi}^u$ given prices $\mathbf{p}^u$. A more general definition for the conditional supplies can then be written as

E1 $$\mathbf{y}(\mathbf{p}, \boldsymbol{\pi}^u) = \left\{(\mathbf{y}^0, \mathbf{y}^u) \in \boldsymbol{\Omega}(\mathbf{p}^u, \boldsymbol{\pi}^u) \,|\mathbf{p}^0 \cdot \mathbf{y}^0 \geq \mathbf{p}^0 \cdot \mathbf{y}^{0\prime} \text{for all } \left(\mathbf{y}^{0\prime}, \mathbf{y}^{u\prime}\right) \in \boldsymbol{\Omega}(\mathbf{p}^u, \boldsymbol{\pi}^u)\right\},$$

with the immediate profit (IP) function

E2 $$\pi_0(\mathbf{p}, \boldsymbol{\pi}^u) = \mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p}, \boldsymbol{\pi}^u).$$

Proposition 1:

P.1 $\mathbf{y}(\mathbf{p}, \boldsymbol{\pi}^u)$ and $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ are homogeneous of degree zero and one in $\mathbf{p}^0$, and both are homogeneous of degree zero in $\mathbf{p}^s$ and π_s^u for *s* = 1,…, *S*;

P.2 $(\mathbf{p}^{0\prime} - \mathbf{p}^0) \cdot \left(\mathbf{y}^0\left((\mathbf{p}^{0\prime}, \mathbf{p}^u), \boldsymbol{\pi}^u\right) - \mathbf{y}^0\left((\mathbf{p}^0, \mathbf{p}^u), \boldsymbol{\pi}^u\right)\right) \geq 0$ for all $\mathbf{p}^{0\prime}, \mathbf{p}^0 \in \mathbb{R}_{++}^L$ and $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ is convex in $\mathbf{p}^0$; and

P.3 $\frac{\partial \pi_0(\mathbf{p}, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s} \leq 0$, $y_l^s(\mathbf{p}', \boldsymbol{\pi}^{u\prime}) = -\frac{\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial p_l^s}}{\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s^u}}$, and $y_l^0(\mathbf{p}', \boldsymbol{\pi}^u) = \frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^u)}{\partial p_l^0}$ for *l* = 1,…,*L* and *s* = 1,…, *S* if $\pi_0(\mathbf{p}, \boldsymbol{\pi}^u)$ is differentiable at $\boldsymbol{\pi}^{u\prime}$ and $\mathbf{p}'$, and $\frac{\partial \pi_0(\mathbf{p}\prime, \boldsymbol{\pi}^{u\prime})}{\partial \pi_s^u} < 0$.

Proof: P.1 follows immediately from equations E1 and E2 given $\alpha\mathbf{p}^0 \cdot \mathbf{y}^0 \geq \alpha\mathbf{p}^0 \cdot \mathbf{y}^{0\prime}$ implies $\mathbf{p}^0 \cdot \mathbf{y}^0 \geq \mathbf{p}^0 \cdot \mathbf{y}^{0\prime}$, and $\alpha\mathbf{p}^s \cdot \mathbf{y}^s \geq \alpha\pi_s^u$ implies $\mathbf{p}^s \cdot \mathbf{y}^s \geq \pi_s^u$ such that $\boldsymbol{\Omega}(\mathbf{p}^u, \boldsymbol{\pi}^u)$ is homogeneous of degree zero in $\mathbf{p}^s$ and π_s^u for s = 1,…,S.

Equations E1 and E2 also imply

E3a $\quad \mathbf{p}^{0\prime} \cdot \mathbf{y}^0(\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u) \geq \mathbf{p}^{0\prime} \cdot \mathbf{y}^0(\mathbf{p}^0, \mathbf{p}^u, \boldsymbol{\pi}^u)$ and

E3b $\quad \mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p}^0, \mathbf{p}^u, \boldsymbol{\pi}^u) \geq \mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u)$

for all $\mathbf{p}^0, \mathbf{p}^{0\prime} \in \mathbb{R}_{++}^L$. Summing these equations and some algebra yields the desired result for the first part of P.2.

For the second part of P.2, convexity in $\mathbf{p}^0$ implies

E4 $\quad \alpha\pi_0(\mathbf{p}^0, \mathbf{p}^u, \boldsymbol{\pi}^u) + (1-\alpha)\pi_0(\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u) \geq \pi_0(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u)$

for all $\mathbf{p}^0, \mathbf{p}^{0\prime} \in \mathbb{R}_{++}^L$ and $\alpha \in [0,1]$. Suppose this is not the case such that there exists $\alpha^0 \in [0,1]$ where

E5 $\quad \alpha^0\pi_0(\mathbf{p}^0, \mathbf{p}^u, \boldsymbol{\pi}^u) + (1-\alpha^0)\pi_0(\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u) < \pi_0(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u)$.

By equations E1 and E2

E6a $\quad \alpha^0\mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p}^0, \mathbf{p}^u, \boldsymbol{\pi}^u) \geq \alpha^0\mathbf{p}^0 \cdot \mathbf{y}^0(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u)$ and

E6b $\quad (1-\alpha^0)\mathbf{p}^{0\prime} \cdot \mathbf{y}^0(\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u) \geq (1-\alpha^0)\mathbf{p}^{0\prime} \cdot \mathbf{y}^0(\alpha\mathbf{p}^0 + (1-\alpha)\mathbf{p}^{0\prime}, \mathbf{p}^u, \boldsymbol{\pi}^u)$.

Summing these equations and substitution using equation E2 then yields the contradiction.

For P.3, note that the producer's problem can also be written as

E7a $\quad \max_{\boldsymbol{y}} \mathbf{p}^0 \cdot \mathbf{y}^0$ subject to $T(\mathbf{y}) = 0$ and $\mathbf{p}^s \cdot \mathbf{y}^s \geq \pi_s^u$ for s = 1,…, S,

which has the Lagrangian

E7b $\quad L = \mathbf{p}^0 \cdot \mathbf{y}^0 - \gamma T(\mathbf{y}) + \sum_{t=1}^S \lambda_t(\mathbf{p}^t \cdot \mathbf{y}^t - \pi_t^u)$

and first order conditions

E7c $\quad \lambda_s{}^* p_l^s - \gamma^* \frac{\partial T(\mathbf{y}^*)}{\partial y_l^s} = 0$ for s = 1,…, S and l = 1,…,L,

E7d $\quad p_l^0 - \gamma^* \frac{\partial T(\mathbf{y}^*)}{\partial y_l^0} = 0$ for l = 1,…,L,

E7e $\quad \mathbf{p}^s \cdot \mathbf{y}^{s*} = \pi_s^u$ for s = 1,…, S, and

E7f $\quad T(\mathbf{y}^*) = 0$.

Now note that

E8 $\pi_0(\mathbf{p},\boldsymbol{\pi}^u) = \mathbf{p}^0 \cdot \mathbf{y}^0(\mathbf{p},\boldsymbol{\pi}^u) - \gamma(\mathbf{p},\boldsymbol{\pi}^u)T\big(\mathbf{y}(\mathbf{p},\boldsymbol{\pi}^u)\big) + \sum_{t=1}^{S}\lambda_t(\mathbf{p},\boldsymbol{\pi}^u)(\mathbf{p}^t \cdot \mathbf{y}^t(\mathbf{p},\boldsymbol{\pi}^u) - \pi_t^u)$

such that the envelope theorem implies $\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial\pi_s^u} = -\lambda_s(\mathbf{p},\boldsymbol{\pi}^u) \leq 0$, $\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial p_l^s} = \lambda_s(\mathbf{p},\boldsymbol{\pi}^u) y_l^s(\mathbf{p},\boldsymbol{\pi}^u)$ or $y_l^s(\mathbf{p},\boldsymbol{\pi}^u) = -\frac{\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial p_l^s}}{\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial\pi_s^u}}$ for $\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial\pi_s^u} < 0$, and $y_l^0(\mathbf{p},\boldsymbol{\pi}^u) = \frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial p_l^0}$ as desired. Q.E.D.

Lemma 2: For $\boldsymbol{\pi}^u(\mathbf{p}) = \operatorname{argmax}_{\boldsymbol{\pi}^u} W(\pi_0(\mathbf{p},\boldsymbol{\pi}^u),\boldsymbol{\pi}^u)$, $\mathbf{y}(\mathbf{p}) = \mathbf{y}\big(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})\big)$.

Proof: Given Lemma 1 and A.5, A.9 can also be written as

E9a $\max_{\mathbf{y}} W(\mathbf{p}^0 \cdot \mathbf{y}^0, \dots, \mathbf{p}^S \cdot \mathbf{y}^S)$ subject to $T(\mathbf{y}) = 0$,

which has the Lagrangian

E9b $L = W(\mathbf{p}^0 \cdot \mathbf{y}^0, \dots, \mathbf{p}^S \cdot \mathbf{y}^S) - \gamma T(\mathbf{y})$

and first order conditions

E9c $p_l^s W_s(\mathbf{p}^0 \cdot \mathbf{y}^0, \dots, \mathbf{p}^S \cdot \mathbf{y}^S) - \gamma^* \frac{\partial T(\mathbf{y}^*)}{\partial y_l^s} = 0$ for s = 0,…, S and l = 1,…,L, and

E9d $T(\mathbf{y}^*) = 0$.

These conditions imply

E9e $\frac{\frac{\partial T(\mathbf{y}^*)}{\partial y_l^s}}{\frac{\partial T(\mathbf{y}^*)}{\partial y_1^0}}\frac{p_1^0}{p_l^s} = \frac{W_s(\mathbf{p}^0\cdot\mathbf{y}^0,\dots,\mathbf{p}^S\cdot\mathbf{y}^S)}{W_0(\mathbf{p}^0\cdot\mathbf{y}^0,\dots,\mathbf{p}^S\cdot\mathbf{y}^S)}$ for s = 0,…, S, l = 1,…,L, and $s \neq 0$ and $l \neq 1$.

The solution to equations E9d and E9e is $\mathbf{y}(\mathbf{p})$.

Now consider the problem

E10a $\max_{\boldsymbol{\pi}^u} W(\pi_0(\mathbf{p},\boldsymbol{\pi}^u),\boldsymbol{\pi}^u)$,

with its associated first order condition

E10b $W_s(\boldsymbol{\pi}^*) + W_0(\boldsymbol{\pi}^*)\frac{\partial\pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial\pi_s} = 0$ for s = 1,...,S.

Note that E7f in the derivation of the IP function is identical to E9d. Also note that equations E7c and E7d, and the proof to P.3 imply

E11 $$-\frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^u)}{\partial \pi_s^u} = \frac{p_1^0}{p_l^s}\frac{\frac{\partial T(\mathbf{y}^*)}{\partial y_l^s}}{\frac{\partial T(\mathbf{y}^*)}{\partial y_1^0}}$$ for s = 0,…, S, l = 1,…,L, and $s \neq 0$ and $l \neq 1$.

Substitution of equation E10b into E11 and some algebra then yields equation E9e. Therefore, the solution to E9a and A.9 must equal the combined solution to equations E1 and E10a as desired. Q.E.D.

Proposition 2: The effect of a change in the price of commodity k = 1,…, L in state t = 0,…, S on the commodity supply $\mathbf{y}(\mathbf{p})$ is

$$\frac{\partial \mathbf{y}(\mathbf{p})}{\partial p_k^t} = \frac{\partial \mathbf{y}\left(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})\right)}{\partial p_k^t} + \frac{\partial \mathbf{y}\left(\mathbf{p},\boldsymbol{\pi}^u(\mathbf{p})\right)}{\partial \boldsymbol{\pi}^u}\frac{\partial \boldsymbol{\pi}^u(\mathbf{p})}{\partial p_k^t}$$

where

$$\frac{\partial \boldsymbol{\pi}^u(\mathbf{p})}{\partial p_k^t} = (\mathbf{P} + \boldsymbol{\Delta\theta}^0 - \mathbf{T}_{\boldsymbol{\pi}^u})^{-\mathbf{1}}\left(\boldsymbol{\Delta}\frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial p_k^t} + \mathbf{T}_{p_k^t}\right) \text{ and}$$

$$\boldsymbol{\theta}^0 = \left(\theta_1^0(\boldsymbol{\pi}(\mathbf{p})) \quad \cdots \quad \theta_S^0(\boldsymbol{\pi}(\mathbf{p}))\right).$$

Proof: The first equation in Proposition 2 is simply an application of the chain rule. For the second equation, rewrite equation E10b as

E12a $$\theta_s^0(\pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*}),\boldsymbol{\pi}^{u^*}) + \frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial \pi_s} = 0$$ for s = 1, …, S

where $\theta_s^0(\boldsymbol{\pi}) = \frac{W_s(\boldsymbol{\pi})}{W_0(\boldsymbol{\pi})}$ and totally differentiate with respect to $\boldsymbol{\pi}^{u^*}$ and p_k^t to get

E12b $$\sum_{r=1}^S\left(\frac{\partial \theta_s^0(\boldsymbol{\pi}^*)}{\partial \pi_0}\frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial \pi_r^u} + \frac{\partial \theta_s^0(\boldsymbol{\pi}^*)}{\partial \pi_r^u} + \frac{\partial^2 \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial \pi_s \partial \pi_r}\right)d\pi_r^u$$

$$+\left(\frac{\partial \theta_s^0(\boldsymbol{\pi}^*)}{\partial \pi_0}\frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial p_k^t} + \frac{\partial^2 \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial \pi_s \partial p_k^t}\right)dp_k^t = 0 \text{ for } s = 1, \ldots, S.$$

Note that

E12c $$\frac{\partial \theta_s^0(\boldsymbol{\pi}^*)}{\partial \pi_0} = \theta_s^0(\boldsymbol{\pi}^*)\left(\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) - \rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right)\right) \text{ and } \frac{\partial \theta_s^0(\boldsymbol{\pi}^*)}{\partial \pi_r^u} = \theta_s^0(\boldsymbol{\pi}^*)\left(\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_r}\right) - \rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_r}\right)\right)$$

where $\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) = -\frac{W_{00}(\boldsymbol{\pi}^*)}{W_0(\boldsymbol{\pi}^*)}$, $\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) = -\frac{W_{s0}(\boldsymbol{\pi}^*)}{W_s(\boldsymbol{\pi}^*)}$, $\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_r}\right) = -\frac{W_{0r}(\boldsymbol{\pi}^*)}{W_0(\boldsymbol{\pi}^*)}$, and $\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_r}\right) = -\frac{W_{sr}(\boldsymbol{\pi}^*)}{W_s(\boldsymbol{\pi}^*)}$.

Equation E12b can then be rewritten as

E12d $$\sum_{r=1}^S\left(\left(\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right) - \rho_s\left(\boldsymbol{\pi}^*,\frac{\partial \boldsymbol{\pi}}{\partial \pi_0}\right)\right)\frac{\partial \pi_0(\mathbf{p},\boldsymbol{\pi}^{u^*})}{\partial \pi_r^u}\right)d\pi_r^u +$$

$$\sum_{r=1}^{S}\left(\left(\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_r}\right)-\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_r}\right)\right)-\frac{\frac{\partial^2\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S\partial\pi_r}}{\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S}}\right)d\pi_r^u =$$

$$-\left(\left(\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)-\rho_s\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)\right)\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial p_k^t}-\frac{\frac{\partial^2\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S\partial p_k^t}}{\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S}}\right)dp_k^t.$$

Also note that

E12e $$\frac{\frac{\partial^2\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S\partial\pi_r}}{\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S}}=\tau_S^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_r}\right) \text{ and } \frac{\frac{\partial^2\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S\partial p_k^t}}{\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial\pi_S}}=\tau_S^{\mathbf{p}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\mathbf{p}}{\partial p_k^t}\right)$$

such that equation E12d becomes

E12f $$(\mathbf{P}+\boldsymbol{\Delta}\boldsymbol{\theta}^0-\mathbf{T}_{\boldsymbol{\pi}^u})\mathbf{d}\boldsymbol{\pi}^u(\mathbf{p})=\left(\boldsymbol{\Delta}\frac{\partial\pi_0\left(\mathbf{p},\boldsymbol{\pi}^{u^*}\right)}{\partial p_k^t}+\mathbf{T}_{p_k^t}\right)dp_k^t$$

where $\mathbf{P}=\begin{bmatrix}\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right)-\rho_1\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)-\rho_1\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)\\ \vdots & \ddots & \vdots \\ \rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right)-\rho_S\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)-\rho_S\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)\end{bmatrix}$, $\boldsymbol{\Delta}=\begin{bmatrix}\rho_1\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)-\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)\\ \vdots \\ \rho_S\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)-\rho_0\left(\boldsymbol{\pi}^*,\frac{\partial\boldsymbol{\pi}}{\partial\pi_0}\right)\end{bmatrix}$, $\mathbf{T}_{\boldsymbol{\pi}^u}=$ $\begin{bmatrix}\tau_1^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \tau_1^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)\\ \vdots & \ddots & \vdots\\ \tau_S^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_1}\right) & \cdots & \tau_S^{\boldsymbol{\pi}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\boldsymbol{\pi}}{\partial\pi_S}\right)\end{bmatrix}$, $\mathbf{T}_{p_k^t}=\begin{bmatrix}\tau_1^{\mathbf{p}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\mathbf{p}}{\partial p_k^t}\right)\\ \vdots\\ \tau_S^{\mathbf{p}}\left(\mathbf{p},\boldsymbol{\pi}^u,\frac{\partial\mathbf{p}}{\partial p_k^t}\right)\end{bmatrix}$ and $\boldsymbol{\theta}^0=[\theta_1^0(\mathbf{p}) \quad \cdots \quad \theta_S^0(\mathbf{p})]$. Some algebraic manipulation of equation E12f then yields the desired result. Q.E.D.