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Network Formation and Financial Fragility

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Abstract

A tractable model of the formation of financial networks is developed, allowing the use of concepts from portfolio theory. The optimal financial network maximizes a Sharpe ratio defined for financial networks, whereas the equilibrium financial network emerges from banks bargaining over future proceeds of co-investment opportunities. Measures of financial fragility, systemic risk and robustness are developed. The equilibrium financial network is shown to be the most connected and with the lowest level of financial fragility, whereas the optimal is the one least connected and with lowest exposure to systemic risk, being also the most robust financial network.

Keywords: Government intervention; financial fragility; financial networks; systemic risk; robustness.

JEL Classification: G1; G2; G3.

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1 Introduction

The importance of having knowledge of the structure of financial networks has become even clearer after the financial crisis of 2007-2009, as pointed out in Yellen (2014). The possibility of contagion and of a systemic crisis following a bad shock in the economy depends, among other things, on the distribution of claims among financial institutions or, in other words, their interconnectedness.

Prior to the crisis, a literature applying networks in the study of financial contagion already existed. The seminal work of Allen and Gale (2000) shows forcibly that network structure matters, as different networks that can implement the social optimal allocation might have inherently different degrees of financial fragility, and hence different levels of exposure to the possibility of contagion[1]. Not only that, models of endogenous network formation were also developed[2].

The paper aims at contributing to the literature on financial network formation, with the ultimate goal of studying financial fragility. The novelty introduced is the use of portfolio theory to characterize optimal financial networks, comparing their properties to those of equilibrium financial networks that would emerge from banks following a Nash bargaining protocol in bilateral meetings.

In a 3 banks framework, any financial network has an implicit measure of expected value and variance of social welfare, defined as the sum of banks' expected utility. From that, a Sharpe ratio for financial networks is calculated, which by definition is maximized under the optimal financial network. The equilibrium financial network, in contrast, is obtained after banks bargain over the future profits of their respective investment opportunities.

The optimal financial network is shown to be the one where banks remain in autarky, i.e., they stay separate and only invest in their own projects. The equilibrium financial network, on the other hand, has maximal connectivity, with banks co-investing in other

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[1]Other papers that also study the association between networks and contagion are Rochet and Tirole (1996), Kiyotaki and Moore (1997), Freixas, Parigi, and Rochet (2000), Eisenberg and Noe (2001), Lagunoff and Schreft (2001), Cifuentes, Ferrucci, and Shin (2005), Nier, Yang, Yorulmazer, and Alentorn (2007), Brusco and Castiglionesi (2007), Caballero and Simsek (2012), Zawadowski (2011), Haldane and May (2011), Duffie (2011) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013), to cite a few.

[2]See Leitner (2005), Babus (2009), Castiglionesi and Navarro (2011), Cohen-Cole, Patacchini, and Zenou (2012) and more recently Farboodi (2014).

banks that have available an investment opportunity with higher expected payoff than their own.

Defining financial fragility as the probability of at least one bank failure, whereas exposure to systemic risk as the probability of a collective failure of the banks in the financial network, the main result of the paper establishes that, on the one hand, the equilibrium financial network has the minimal level of financial fragility and, on the other hand, the optimal financial network has the minimal level of exposure to systemic risk. The robustness of a financial network, defined as the probability of a collective failure of banks given that at least one bank in the financial network fails, summarizes the trade-off between fragility and systemic risk. The optimal financial network is shown to achieve the maximum level of robustness.

Despite using a completely different framework, the results of the paper are similar to the ones in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013). Acemoglu et al. show that the relationship between the possibility of contagion and the structure of the financial network depends on the magnitude of negative shocks: more connected structures are more robust in face of small shocks and more fragile in the case of large ones. In the present paper, there are only large shocks (payoff of investments are either positive or zero), and the more robust financial network is also the one with less connections, namely the one with banks in autarky.

The structure of the paper is as follows: Section 2 details the model; Section 3 defines optimal financial networks; Section 4 discusses about financial fragility; finally, Section 5 contains the concluding remarks and some venues for future research.

2 Model

Consider a one good, two dates $t = 0, 1$ economy, consisting of three separate regions $i \in \{1, 2, 3\}$, each of them with a representative bank $B_i \in \{B_1, B_2, B_3\}$. Each bank is endowed with two units of the good at date $t = 0$ but nothing at $t = 1$, i.e.:

$$e_i = (2, 0), \tag{1}$$

where the first coordinate of e_i represents bank's i endowment of the good at $t = 0$ and the second at $t = 1$.

Banks have risk-neutral preferences given by:

$$U_i(W_i) = W_i, \tag{2}$$

where W_i represents the wealth of bank i at date $t = 1$.

Each bank dispose of a production technology such that, by investing I at $t = 0$, bank i gets at $t = 1$:

$$T_i(I) = \begin{cases} IR_i, & \text{with probability} \quad p_i, \\ 0, & \text{with probability} \quad 1 - p_i. \end{cases} \tag{3}$$

The ordering of banks is:

$$R_1 > R_2 > R_3 \quad \text{and} \quad p_1 < p_2 < p_3, \tag{4}$$

i.e., the higher the payoff per unit of investment, the less likely the production technologies are to succeed.

From (3), it follows that the expected payoff per unit of investment and variance of the production technologies, μ_i and σ_i, respectively, are:

$$\mu_i = p_i R_i, \tag{5}$$
$$\sigma_i = p_i(1 - p_i) R_i^2. \tag{6}$$

The payoff of the different production technologies is taken to be independently distributed across banks. Also, it is assumed that the higher the expected payoff of the production technology, the higher the risk (variance) entailed by it, whereas the ratio of the two remains constant. Equivalently:

$$p_1 R_1 > p_2 R_2 > p_3 R_3, \tag{7}$$
$$p_1(1 - p_1) R_1^2 > p_2(1 - p_2) R_2^2 > p_3(1 - p_3) R_3^2, \tag{8}$$
$$(1 - p_1) R_1 = (1 - p_2) R_2 = (1 - p_3) R_3. \tag{9}$$

At date $t = 0$ banks meet pairwise, occasion in which they can borrow or lend one unit of the good to each other. Let χ_{ij} be the variable defining if banks i and j are connected, as the following:

$$\chi_{ij} := \begin{cases} 1 & \text{if } i \text{ lends to } j, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

A *financial network* is defined as a vector $(\chi_{12}, \chi_{13}, \chi_{21}, \chi_{23}, \chi_{31}, \chi_{32})$ such that χ_{ij} is defined as in (10). Figure 1 depicts all the possible financial networks that could be formed by the three banks in the economy, grouped according to the similarities of the borrowing and lending activities of each bank. An arrow going from bank i to bank j, to be called a *link*, exists if and only if $\chi_{ij} = 1$.
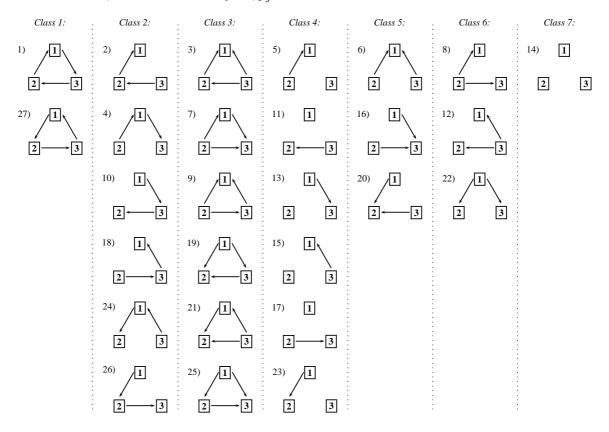

Figure 1: Space of financial networks.

2.1 Bargaining and Formation of Financial Networks

As mentioned in the previous section, at date $t = 0$ banks meet pairwise and decide whether or not to borrow or lend to each other. Consider two arbitrary banks, i and j. If bank i agrees to lend to bank j, then the production technology of the later yields $\nu_{i \to j} = T_j(1)$, as in (3). Without agreement, bank i uses the good on its own production technology and gets $\nu_{i,i \not\to j} = T_i(1)$, whereas bank j's payoff is zero, $\nu_{j,i \not\to j} = 0$. This characterizes[3] a two-person bargaining problem with transferable utility where $\nu_{i,i \not\to j}$ is

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[3]See Chapter 8 in Myerson (1991).

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the disagreement payoff to player (bank) i; $\nu_{j,i \nrightarrow j}$ is the disagreement payoff to player (bank) j; and $\nu_{i \rightarrow j}$ is the total transferable wealth available to players if they cooperate. The *Nash bargaining solution* of this game is the date $t = 1$ allocation vector $(\phi_{i,i \rightarrow j}, \phi_{j,i \rightarrow j})$ where:

$$\phi_{i,i \rightarrow j} = \frac{\nu_{i \rightarrow j} + \nu_{i,i \nrightarrow j} - \nu_{j,i \nrightarrow j}}{2} = \frac{T_j(1) + T_i(1)}{2}, \tag{11}$$

$$\phi_{j,i \rightarrow j} = \frac{\nu_{i \rightarrow j} + \nu_{j,i \nrightarrow j} - \nu_{i,i \nrightarrow j}}{2} = \frac{T_j(1) - T_i(1)}{2}. \tag{12}$$

Since interbank lending requires an agreement at date $t = 0$, in order to decide whether or not to either borrow or lend banks instead have to consider what the Nash bargaining solution yields to them in expected terms, $\left(\overline{\phi}_{i,i \rightarrow j}, \overline{\phi}_{j,i \rightarrow j}\right)$:

$$\overline{\phi}_{i,i \rightarrow j} = \frac{\mu_j + \mu_i}{2}, \tag{13}$$

$$\overline{\phi}_{j,i \rightarrow j} = \frac{\mu_j - \mu_i}{2}. \tag{14}$$

Therefore, the Nash bargaining solution implies that a necessary and sufficient condition for a loan to exist between bank i (lender) and bank j (borrower) is:

$$\overline{\phi}_{j,i \rightarrow j} > 0 \Leftrightarrow \mu_j > \mu_i \Leftrightarrow p_j R_j > p_i R_i, \tag{15}$$

which is to be called the *participation constraint*. A *feasible* financial network is one where the participation constraint implied by every link is satisfied.

In a financial network $S = (\chi_{12}, \chi_{13}, \chi_{21}, \chi_{23}, \chi_{31}, \chi_{32})$ representing banks' meetings and interbank lending decisions, the payoff of each bank at date $t = 1$ is given by:

$$\nu_S(i) = \sum_{j \neq i} \left[\chi_{ij} \phi_{i,i \rightarrow j} + (1 - \chi_{ij}) \phi_{i,i \nrightarrow j} + \chi_{ji} \phi_{i,j \rightarrow i} \right], \tag{16}$$

and the expected payoff at date $t = 0$ analogously by:

$$\overline{\nu}_S(i) = \sum_{j \neq i} \left[\chi_{ij} \overline{\phi}_{i,i \rightarrow j} + (1 - \chi_{ij}) \overline{\phi}_{i,i \nrightarrow j} + \chi_{ji} \overline{\phi}_{i,j \rightarrow i} \right], \tag{17}$$

with $\nu_S = (\nu_S(1), \nu_S(2), \nu_S(3))$ and $\overline{\nu}_S = (\overline{\nu}_S(1), \overline{\nu}_S(2), \overline{\nu}_S(3))$.

A financial network $\widehat{S} = (\widehat{\chi}_{12}, \widehat{\chi}_{13}, \widehat{\chi}_{21}, \widehat{\chi}_{23}, \widehat{\chi}_{31}, \widehat{\chi}_{32})$ is then said to be an *equilibrium* if and only if there is no other feasible financial network such that all its members are as well-off and at least one is strictly better, i.e.:

$$\left\{ \left(\chi_{12}, \chi_{13}, \chi_{21}, \chi_{23}, \chi_{31}, \chi_{32} \right) \mid \overline{\nu}_S > \overline{\nu}_{\widehat{S}} \right\} = \emptyset, \tag{18}$$

where $\overline{\nu}_S > \overline{\nu}_{\widehat{S}}$ if and only if $\overline{\nu}_S(i) \geq \overline{\nu}_{\widehat{S}}(i)$ and the inequality holds strictly for at least one i.

By adopting a purely utilitarian approach, the *social welfare* $W(\nu_S)$ implied by a financial network S is given by:

$$W(\nu_S) = \sum_i \nu_S(i), \tag{19}$$

which after substituting for (16) and lengthy manipulations can be written as:

$$W(\nu_S) = \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] T_i(1). \tag{20}$$

Accordingly, the expected value and variance of social welfare are given by:

$$\mu_S = \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] \mu_i, \tag{21}$$

$$\sigma_S = \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right]^2 \sigma_i, \tag{22}$$

where it is used the assumption that banks have independently distributed production technologies.

3   Optimal Financial Networks

Characterizing financial networks by their expected value and variance of social welfare, (21) and (22), respectively, suggests a measure of efficiency as the one pioneered in Markowitz (1952). Accordingly, a financial network is called *efficient* if it yields the maximum expected social welfare for a specific level of variance or, analogously, if it yields the lowest variance for a specific level of expected social welfare. The following proposition establishes that, in the current setup, every feasible financial network lies in the efficient frontier:

Proposition 1. *Any feasible financial network S is efficient.*

Proof. Let $S = (\chi_{ij})_{i,j=1,2,3}$ and $S' = \left(\chi'_{ij}\right)_{i,j=1,2,3}$ be two arbitrary and different feasible financial networks, i.e., both satisfy the participation constraints and there exists at least one pair of banks i and j for which $\chi_{ij} \neq \chi'_{ij}$. The assertion of the proposition is equivalent to the statement that $\mu_S > \mu_{S'} \iff \sigma_S > \sigma_{S'}$, and substituting for (21) and (22) it follows that:

$$
\begin{aligned}
& \mu_S > \mu_{S'} \\
\iff & \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] \mu_i > \sum_i \left[2 + \sum_{j \neq i} \left(\chi'_{ji} - \chi'_{ij}\right) \right] \mu_i \\
\iff & \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] \sigma_i > \sum_i \left[2 + \sum_{j \neq i} \left(\chi'_{ji} - \chi'_{ij}\right) \right] \sigma_i,
\end{aligned} \tag{23}
$$

where the last inequality uses $\mu_i = \sigma_i \left(\mu_i/\sigma_i\right)$ and the assumption that $\mu_i/\sigma_i = \mu_j/\sigma_j$ for any i and j, as in (9). Finally, using the assumption expressed in (8) and since $2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \geq 0$ for any $i = 1, 2, 3$, from (23) it follows that:

$$
\begin{aligned}
& \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] \sigma_i > \sum_i \left[2 + \sum_{j \neq i} \left(\chi'_{ji} - \chi'_{ij}\right) \right] \sigma_i \\
\iff & \sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right]^2 \sigma_i > \sum_i \left[2 + \sum_{j \neq i} \left(\chi'_{ji} - \chi'_{ij}\right) \right]^2 \sigma_i \\
\iff & \sigma_S > \sigma_{S'},
\end{aligned} \tag{24}
$$

which completes the proof. ∎

Therefore, if a financial network S have a higher expected social welfare than a financial network S', the variance of S will be correspondingly higher than the variance of S', and vice versa. As long as there is a well defined trade-off between expected value and variance, or a selection criterium for financial networks in the efficient frotier, this is not a useful result. A way around that is the *Sharpe ratio*, a measure that combines the expected value and variance of social welfare of a financial network S, defined as:

$$
sr\left(S\right) = \frac{\mu_S}{\sqrt{\sigma_S}} = \frac{\sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right] \mu_i}{\sqrt{\sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij}) \right]^2 \sigma_i}}. \tag{25}
$$

In this context, a financial network $S^* = (\chi^*_{12}, \chi^*_{13}, \chi^*_{21}, \chi^*_{23}, \chi^*_{31}, \chi^*_{32})$ is said to be *optimal* if it is feasible and maximizes the Sharpe ratio, i.e.:

$$S^* \in \{S = (\chi_{12}, \chi_{13}, \chi_{21}, \chi_{23}, \chi_{31}, \chi_{32}) \mid sr\,(S') \leq sr\,(S), \forall S'\}. \tag{26}$$

Under the assumptions expressed by (7), (8) and (9), the next two propositions constitute the main results of the paper:

Proposition 2. *The unique equilibrium financial network is:*

$$\widehat{S} = (\widehat{\chi}_{12}, \widehat{\chi}_{13}, \widehat{\chi}_{21}, \widehat{\chi}_{23}, \widehat{\chi}_{31}, \widehat{\chi}_{32}) = (0, 0, 1, 0, 1, 1). \tag{27}$$

Proof. By definition, an equilibrium financial network must be feasible, which means that every link (if it exists) has to satisfy a participation constraint. Hence, interbank lending between banks i (lender) and j (borrower) is allowed only if $\mu_j > \mu_i$. This proves that $\chi_{ij} = 1 \Rightarrow \mu_j > \mu_i$. Now, recall that, for a financial network to be an equilibrium, there cannot be any other feasible network such that every bank is as well-off and at least one bank is strictly better. Suppose then that S' is an equilibrium financial network such that $\mu_j > \mu_i$ and $\chi'_{ij} = 0$, for arbitrary i and j. Consider now the financial network S'' that is obtained from S' by adding the link $\chi''_{ij} = 1$. From the Nash bargaining solution given by (13) and (14), the net gains of banks i and j by switching from S' to S'' are:

$$\overline{\phi}_{i,i\to j} - \overline{\phi}_{i,i\nrightarrow j} = \frac{\mu_j + \mu_i}{2} - \mu_i > 0, \tag{28}$$

$$\overline{\phi}_{j,i\to j} - \overline{\phi}_{j,i\nrightarrow j} = \frac{\mu_j - \mu_i}{2} - 0 > 0, \tag{29}$$

contradicting the assumption that S' is an equilibrium network, which proves that $\mu_j > \mu_i \Rightarrow \chi_{ij} = 1$.

∎

Proposition 3. *The equilibrium financial network is not optimal.*

Proof. By definition, the optimal financial network S^* maximizes the Sharpe ratio defined in (25), i.e., it is the solution to:

$$\max_{(\chi_{ij})_{i,j=1,2,3}} sr\,(S) = \frac{\sum_i \left[2 + \sum_{j\neq i} (\chi_{ji} - \chi_{ij})\right] \mu_i}{\sqrt{\sum_i \left[2 + \sum_{j\neq i} (\chi_{ji} - \chi_{ij})\right]^2 \sigma_i}}. \tag{30}$$

The FOCs with respect to χ_{ij} can be written as:

$$\frac{\mu_i/\sigma_i}{\left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij})\right]} = \frac{\sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij})\right] \mu_i}{\sqrt{\sum_i \left[2 + \sum_{j \neq i} (\chi_{ji} - \chi_{ij})\right]^2 \sigma_i}}, i, j = 1, 2, 3, \qquad (31)$$

which in turn implies that:

$$\frac{\mu_1/\sigma_1}{\left[2 + \sum_{j \neq 1} (\chi_{j1} - \chi_{1j})\right]} = \frac{\mu_2/\sigma_2}{\left[2 + \sum_{j \neq 2} (\chi_{j2} - \chi_{2j})\right]} = \frac{\mu_3/\sigma_3}{\left[2 + \sum_{j \neq 3} (\chi_{j3} - \chi_{3j})\right]}. \qquad (32)$$

By assumption, $\mu_1/\sigma_1 = \mu_2/\sigma_2 = \mu_3/\sigma_3$, and therefore at the optimal financial network S^* it must hold that:

$$\left[2 + \sum_{j \neq 1} \left(\chi_{j1}^* - \chi_{1j}^*\right)\right] = \left[2 + \sum_{j \neq 2} \left(\chi_{j2}^* - \chi_{2j}^*\right)\right] = \left[2 + \sum_{j \neq 3} \left(\chi_{j3}^* - \chi_{3j}^*\right)\right]. \qquad (33)$$

The set of equalities in (33) can equivalently be written as:

$$\chi_{21}^* - \chi_{12}^* = \chi_{13}^* - \chi_{31}^* = \chi_{32}^* - \chi_{23}^*. \qquad (34)$$

Since an optimal financial network is necessarily feasible, from the participation constraints it follows that $\chi_{12}^* = \chi_{13}^* = \chi_{32}^* = 0$, and the set of equalities in (34) is further simplified to $\chi_{21}^* = -\chi_{31}^* = \chi_{32}^*$, which holds if and only if $\chi_{21}^* = \chi_{31}^* = \chi_{32}^* = 0$. Since the equilibrium financial network has $\widehat{\chi}_{21} = \widehat{\chi}_{31} = \widehat{\chi}_{32} = 1$, it cannot be optimal.

∎

The proof of Proposition 3 indicates that the optimal financial network is the one where banks do not share any links, i.e., $\chi_{ij}^* = 0$, for $i, j = 1, 2, 3$, which is precisely financial network 14) in Figure 1.

Despite being not optimal, the equilibrium financial network is the one with both highest expected value of social welfare and variance, a result that follows immediately from Proposition 1 above:

Corollary 4. *The equilibrium financial network $\widehat{S}$ maximizes both the expected value and variance of social welfare.*

Proof. For any arbitrary feasible financial network $S = (\chi_{ij})_{i,j=1,2,3}$, the expected value of social welfare is given by:

$$\begin{aligned} \mu_S &= \left[2 + \sum_{j \neq 1}\left(\chi_{j1} - \chi_{1j}\right)\right]\mu_1 + \left[2 + \sum_{j \neq 2}\left(\chi_{j2} - \chi_{2j}\right)\right]\mu_2 + \left[2 + \sum_{j \neq 3}\left(\chi_{j3} - \chi_{3j}\right)\right]\mu_3 \\ &= \left[2 + \chi_{21} + \chi_{31}\right]\mu_1 + \left[2 + \chi_{32} - \chi_{21}\right]\mu_2 + \left[2 - \chi_{31} - \chi_{32}\right]\mu_3, \quad (35) \end{aligned}$$

where the last equality uses the participation constraints of feasible networks, i.e., $\chi_{12} = \chi_{13} = \chi_{23} = 0$. From the assumption that $\mu_1 > \mu_2 > \mu_3$, expression (35) implies that the maximum expected social welfare is achieved when $\chi_{21} = \chi_{31} = \chi_{32} = 1$, corresponding precisely to the links of the equilibrium financial network. Finally, since from Proposition 1 it holds that $\mu_S > \mu_{S'} \iff \sigma_S > \sigma_{S'}$ and the equilibrium financial network $\widehat{S}$ maximizes the expected value of social welfare, it automatically maximizes variance.

∎

4   Financial Fragility

The present section addresses the implications of the maximizing behavior of banks to the fragility of financial networks and their exposure to systemic risk. As it will be shown, the equilibrium and the optimal financial networks involve a trade-off between fragility and exposure to systemic risk, as defined next.

The *fragility* of a financial network is defined as the probability of a bank failure at date $t = 1$, when the uncertainty about the success or failure of banks' production technologies is resolved. *Systemic risk* is defined as the probability that all the banks in the financial network fail together. The event of a *bank failure* is defined to occur whenever the net worth (assets minus liabilities) or, equivalently, the payoff of a bank defined in (16), is non-positive.

Expression (16) in conjunction with (11) and (12) implies that the net worth of banks in a particular financial network depends on the joint distribution of the random variables representing the production technologies given by (3). With S_i denoting $T_i(I) = IR_i$ and F_i corresponding to $T_i(I) = 0$, the state space of the production technologies is:

$$\Omega = \{(\omega_1, \omega_2, \omega_3) \,|\, \omega_i \in \{S_i, F_i\}, i = 1, 2, 3\}. \quad (36)$$

Accordingly, given that the production technologies are independently distributed, the density function P of any arbitrary $\omega = \{\omega_1, \omega_2, \omega_3\}$ is:

$$P(\omega) = \Pi_i p(\omega_i), \tag{37}$$

where $p(\omega_i) = p_i$ if $\omega_i = S_i$ and $1 - p_i$ whenever $\omega_i = F_i$.

Finally, a function $f_S : \Omega \mapsto \{0, 1, 2, 3\}$ can be defined to designate the number of bank failures in a financial network S associated to each element from the state space of production technologies. Let $D(\omega; S) = \{i \,|\, \nu_S(i) \le 0, i = 1, 2, 3\}$ be the set containing all the banks with non-positive payoff at date $t = 1$ after an arbitrary $\omega \in \Omega$, with $\nu_S(i)$ defined in (16). The function f_S is then given by:

$$f_S(\omega) = |D(\omega; S)|, \tag{38}$$

where $|\cdot|$ is the cardinality operator. On the other hand, the set $f_S^{-1}(k)$, defined by:

$$f_S^{-1}(k) = \{\omega \in \Omega \,|\, f_S(\omega) = k\}, \tag{39}$$

contains all the elements of the state space leading to a number of bank failures in financial network S equal to k, for $k \in \{0, 1, 2, 3\}$, and its measure $M_S(k)$ equals to:

$$M_S(k) = \sum_{\omega \in f_S^{-1}(k)} P(\omega). \tag{40}$$

A financial network S, therefore, has fragility φ_S and exposure to systemic risk r_S formally given by:

$$\varphi_S = 1 - M_S(0), \tag{41}$$

$$\eta_S = M_S(3). \tag{42}$$

The following proposition establishes the trade-off between financial fragility and exposure to systemic risk entailed by the equilibrium and the optimal financial networks:

Proposition 5. *The equilibrium financial network $\widehat{S}$ has lower financial fragility than the optimal financial network S^*. Conversely, the optimal financial network S^* has lower exposure to systemic risk than the equilibrium financial network $\widehat{S}$.*

Proof. Based on (16), the net worth of banks in the equilibrium financial network $\widehat{S}$ is given by:

$$\nu_{\widehat{S}}(1) \;=\; 2T_1(1) + \frac{T_1(1) - T_2(1)}{2} + \frac{T_1(1) - T_3(1)}{2}, \tag{43}$$

$$\nu_{\widehat{S}}(2) \;=\; \frac{T_1(1) + T_2(1)}{2} + T_2(1) + \frac{T_2(1) - T_3(1)}{2}, \tag{44}$$

$$\nu_{\widehat{S}}(3) \;=\; \frac{T_1(1) + T_2(1)}{2} + \frac{T_2(1) + T_3(1)}{2}, \tag{45}$$

whereas in the optimal financial network S^* it is:

$$\nu_{S^*}(1) \;=\; 2T_1(1), \tag{46}$$

$$\nu_{S^*}(2) \;=\; 2T_2(1), \tag{47}$$

$$\nu_{S^*}(3) \;=\; 2T_3(1), \tag{48}$$

where $T_i(I)$ is the random variable that represents the production technology defined in (3). These, together with (41) and (42), imply that:

$$\varphi_{\widehat{S}} = 1 - p_1 \quad \text{and} \quad \eta_{\widehat{S}} = (1 - p_1)(1 - p_2), \tag{49}$$

$$\varphi_{S^*} = 1 - p_1 p_2 p_3 \quad \text{and} \quad \eta_{S^*} = (1 - p_1)(1 - p_2)(1 - p_3), \tag{50}$$

hence $\varphi_{\widehat{S}} < \varphi_{S^*}$ and $\eta_{S^*} < \eta_{\widehat{S}}$.

∎

By computing the measures of financial fragility and exposure to systemic risk for each of the feasible financial networks, it can be shown that, in fact, the equilibrium financial network $\widehat{S}$ achieves the lowest bound of financial fragility whereas the optimal financial network S^* achieves the lowest bound of exposure to systemic risk.

From a police perspective, therefore, the choice for the equilibrium or the optimal financial network will depend on whether financial fragility or rather exposure to systemic risk is deemed more important. One possible way to incorporate these two concepts is by defining a measure of robustness of financial networks. Accordingly, the *robustness* ρ_S of a financial network S is defined as the probability of not having a systemic crisis (failure of all the banks) conditional on the event that at least one bank fails, i.e.:

$$
\begin{aligned}
\rho_S &= 1 - P\left(f_S\left(\omega\right) = 3 \mid f_S\left(\omega\right) > 0\right) \\
&= 1 - \frac{M_S\left(3\right)}{1 - M_S\left(0\right)} \\
&= 1 - \frac{\eta_S}{\varphi_S},
\end{aligned}
\tag{51}
$$

where the functions f_S and M_S are given by (38) and (40), respectively. With such a measure of robustness, as a corollary to the proof of Proposition 5 it follows that:

Corollary 6. *The optimal financial network S^* is more robust than the equilibrium financial network $\widehat{S}$, i.e., $\rho_{S^*} > \rho_{\widehat{S}}$.*

Proof. Follows directly from applying (49) and (50) into the definition of ρ_S. ∎

In fact, by computing the robustness measure for each feasible network, it can be shown that the optimal financial network S^* achieves the upper bound level of robustness, whereas the equilibrium financial network $\widehat{S}$ achieves the lower bound.

5 Concluding Remarks

This paper develops a tractable model of the formation of financial networks. The main novelty is the use of the Sharpe ratio in the characterization of optimal financial networks. In contrast, equilibrium financial networks are defined as the ones emerging from banks following a Nash bargaining protocol over potential gains from investment opportunities.

The equilibrium financial network is shown to be the most connected, whereas the optimal financial network has banks in autarky, or operating separately. Financial fragility, or the probability of at least one bank failure, is minimized under the equilibrium financial network; the probability of an episode of collective bank failure, defined as exposure to systemic risk, is on the other hand minimized under the optimal financial network. Combining these two definitions leads to a measure of robustness, which is the probability of all the banks failing conditional on the episode of at least one bank failure. The optimal financial network is shown to be the more robust the its equilibrium counterpart.

The main result of the paper, which is the trade-off between financial fragility and exposure to systemic risk expressed by the equilibrium and optimal financial networks, respectively, is similar to the one in Acemoglu, Ozdaglar, and Tahbaz-Salehi (2013), where

the association between the possibility of contagion and the structure of financial networks obeys a phase diagram: more connected network structures are more robust for small levels of negative shocks, but on the other hand expose the system to more fragility as shocks get more negative.

The model can be extended in many different ways. The number of banks can be easily increased without affecting the main results of the paper, as long as the assumption that banks can only co-invest one unit of the good. Removing this assumption might lead resources to be even more concentrated since under risk neutrality banks would co-invest as much as they could in the bank offering an investment opportunity with the highest expected payoff.

The more interesting extension would be to add the possibility of government intervention following the resolution of uncertainty. Upon government intervention, taking the form of either a bailout of a particular bank or as a subsidy to banks' investments, it can be the case that co-investment agreements will take place in situations where otherwise they would be absent, which will change the structure of the financial network formed. Hence, the financial networks obtained like this will have different levels of fragility, exposure to systemic risk and robustness and, therefore, the effects from the possibility of government intervention could directly be seen.

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