AgEcon SEARCH
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

The World's Largest Open Access Agricultural & Applied Economics Digital Library

This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.

Help ensure our sustainability.

Give to AgEcon Search

AgEcon Search
http://ageconsearch.umn.edu
aesearch@umn.edu

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

Integrating Efficiency Concepts in Technology Approximation: A Weighted DEA Approach

Kota Minegishi

University of Maryland, College Park

Contact: kminegishi@arec.umd.edu

Selected Paper prepared for presentation at the Agricultural & Applied Economics Associations 2014 AAEA Annual Meeting, Minneapolis, MN, July 27-29, 2014

Copyright 2014 by Kota Minegishi. All rights reserved. Readers may make verbatim copies of this document for non-commercial purposes by any means, provided that this copyright notice appears on all such copies.

Integrating Efficiency Concepts in Technology Approximation: A Weighted DEA Approach

Kota Minegishi*

Department of Agricultural & Resource Economics,
University of Maryland, College Park

Version. May 19, 2014

Abstract: A method is developed to integrate the efficiency concepts of technical, allocative, and scale inefficiencies (TI, AI, SI) into the variable returns to scale (VRS) frontier approximation in Data Envelopment Analysis (DEA). The proposed weighted DEA (WDEA) approach takes a weighted average of the profit, constant returns to scale (CRS), and VRS frontiers, so that the technical feasibility of a VRS frontier is extended toward scale- and allocatively-efficient decisions. A weight selection rule is constructed based on the empirical performance of the VRS estimator via the local confidence interval of Kneip, Simar, and Wilson (2008). The resulting WDEA frontier is consistent and more efficient than the VRS frontier under the maintained properties of a data generating process. The potential estimation efficiency gain arises from exploiting sample correlations among TI, AI, and SI. Application to Maryland dairy production data finds that technical efficiency is on average 5.2% to 7.8% lower under the WDEA results than under the VRS counterparts.

1 Introduction

The concept of optimality and the subsequent definition of inefficiency depend on the focus of benchmarking by a relevant "frontier" of decision possibilities. In the tradition of production economics, three concepts of optimality stand out. Technical inefficiency (TI) assesses the extent of feasible output expansions for given inputs (or input reductions for given outputs) relative to the technological frontier of input-output decisions. Allocative and scale inefficiencies (AI and SI) represent the extents of forgone opportunities by the misallocation of resources and the suboptimal scales of operations respectively, relative to the frontier of revenue maximization (or cost minimization) and the frontier of linear-homogeneous production process (i.e. constant returns to scale; CRS). Numerous empirical studies have analyzed TI while paying little attention to AI or SI.

*This study is a chapter of the author's dissertation at University of Maryland, College Park. I thank professor Robert Chambers for overseeing the project as well as professor Erik Lichtenberg and Dr. Jim Hanson for helpful comments. I am grateful to Mr. Dale Johnson for sharing his data for this study. All remaining errors are my own. Contact: kota0403@umd.edu

However, the interconnections among the concepts of TI, AI, and SI suggest an opportunity to improve technological frontier estimations. Conceptually, TI is a gap between an input-output decision and a technological frontier, and AI and SI are the gaps between the technological frontier and its outer frontiers of different benchmarking focuses. Empirically, the pivotal role of a technological frontier implies that the most efficient estimation strategy entails a joint specification of the frontier and these inefficiency concepts. In the parametric frontier literature, such simultaneous estimations have been developed mainly by incorporating AI into the optimal factor demands from cost minimization (e.g., Yotopoulos and Lau, 1973; Schmidt and Lovell, 1979, 1980; Kumbhakar, 1989, 1997; Kumbhakar and Wang, 2006; Kumbhakar and Tsionas, 2011). For nonparametric frontier models like Data Envelopment Analysis (DEA), on the other hand, there is no coherent estimation technique that integrates these efficiency concepts. This gap in knowledge is partially filled in this article.

Inefficient DEA estimations manifest themselves in the form of a limited ability to discriminate individual TI measurements. Efficiency analysis with a small sample size tends to find an unexpectedly large number of observations being fully technically-efficient, a pervasive concern in the nonparametric frontier literature (e.g., Dyson et al., 2001; Podinovski and Thanassoulis, 2007). One strand of literature tackles this issue by applying direct value judgments (i.e. shadow price restrictions) based on perceived importance of inputs and outputs (e.g., Allen et al., 1997; Thanassoulis, Portela, and Allen, 2004) or so-called assurance regions (e.g., Dyson and Thanassoulis, 1988; Thompson et al., 1990; Sarrico and Dyson, 2004; Podinovski, 2004a; Tracy and Chen, 2004; Khalili et al., 2010). Other lines of research incorporate additional knowledge on production processes or constrain the range of technological parameters so as to increase estimation efficiency. Examples include weak disposability of inputs or undesirable outputs (Chung, Fre, and Grosskopf, 1997; Scheel, 2001; Seiford and Zhu, 2002; Kuosmanen, 2005; Podinovski and Kuosmanen, 2011), non-discretionary factors (Ruggiero, 1998), unobserved decisions (Thanassoulis and Allen, 1998; Allen and Thanassoulis, 2004), selective linear homogeneity (Podinovski, 2004b; Podinovski and Thanassoulis, 2007), and prescribed producer trade-offs (Podinovski, 2004c). Following the second strand of literature, this article refines a variable returns to scale (VRS) frontier estimation by calibrating the degrees of technical substitution and linear homogeneity, based on sample-level properties of AI and SI respectively. The method is a variant of the DEA frontier bounds of Chambers, Chung, and Färe (1998) and closely related to the allocative inefficiency bounds of Kuosmanen and Post (2001).

Namely, this study proposes a weighted DEA (WDEA) approach that estimates a technological frontier as a weighted average of the profit, CRS, and VRS frontiers. By integrating the

concepts of TI, AI, and SI, it enhances the discriminatory power of DEA. An optimal weight selection rule is devised based on the empirical performance of the VRS estimator via the local confidence interval proposed by Kneip, Simar, and Wilson (2008). The resulting WDEA frontier is consistent and more efficient than the VRS frontier under the maintained properties of a data generating process. The potential estimation efficiency gain arises from exploiting sample correlations among TI, AI, and SI.

In a single-input single-output (x-y) space, figure 1 illustrates the concept of WDEA for the relationships among the CRS, VRS and postulated technological frontiers (depicted as a solid-curve). The optimal projections of the decision at point A to the VRS and CRS frontiers are shown at points B and C, yielding the conventional measures of TI and SI as distances AB and BC respectively. WDEA postulates a technological frontier through a weighted average of the VRS and CRS frontiers (i.e., somewhere between the inner and outer frontier-approximations). The new TI and SI measurements under WDEA are distances $AD(> AB)$ and $DC(< BC)$ where point D denotes the projection of point A onto the WDEA technological frontier. A parallel refinement of the frontier approximation can be obtained using similar relationships among the profit, VRS, and postulated frontiers. Together, these refinements are formalized under a weighted-average of the profit, CRS, and VRS frontiers.

In the following, section 2 presents the WDEA approach, and section 3 applies the method to Maryland dairy production data, followed by conclusions in section 4.

2 Method

In below, technical, allocative, and scale inefficiency (TI, AI, and SI) measurements are denoted by the directional distance function of Chambers, Chung, and Färe (1998). Its additive nature is notationally well-suited for describing the weighted average of these inefficiency concepts. The section consists of the descriptions of preliminary concepts, a weighted DEA (WDEA) approach, and a weight selection for WDEA.

2.1 Preliminaries

Notations and preliminary concepts are defined as follows. Technology T is a set of feasible input-output bundles, or $T = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}_+^L \times \mathbb{R}_+^M : \ \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}$ where

A.1 T is closed.

A.2 T satisfies free-disposability: $(\boldsymbol{x}, \boldsymbol{y}) \in T$ and $(-\boldsymbol{x}, \boldsymbol{y}) \geq (-\boldsymbol{x}', \boldsymbol{y}') \ \Rightarrow \ (\boldsymbol{x}', \boldsymbol{y}') \in T$.

A.3 T is convex: $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{x}', \boldsymbol{y}') \in T \ \Rightarrow \forall \lambda \in [0, 1], \forall (\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{x}', \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{y}') \in T$.

The boundary of a technology is referred to as *technological frontier*. T can be completely characterized by the directional distance function of Chambers, Chung, and Färe (1998)[1] in the sense that $(\boldsymbol{x}, \boldsymbol{y}) \in T \Leftrightarrow D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) \geq 0$ where

$$D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b \in \mathbb{R} : (\boldsymbol{x} - b\boldsymbol{g}_x, \boldsymbol{y} + b\boldsymbol{g}_y) \in T\}. \tag{1}$$

Function $D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)$ measures the distance between point $(\boldsymbol{x}, \boldsymbol{y})$ and the frontier of technology T in direction $(-\boldsymbol{g}_x, \boldsymbol{g}_y)$, representing technical inefficiency (TI). As a special case, setting direction $(-\boldsymbol{g}_x, \boldsymbol{g}_y) = (-\boldsymbol{x}_0, \boldsymbol{0})$ yields an input-oriented, radial TI measurement, which is equivalent to Shephard's input distance function $\theta_V(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max\{\theta : \boldsymbol{x}_0/\theta \in V(\boldsymbol{y}_0)\} \geq 1$ for the input set $V(\boldsymbol{y})$ associated with technology T. Similarly, setting direction $(-\boldsymbol{g}_x, \boldsymbol{g}_y) = (\boldsymbol{0}, \boldsymbol{y}_0)$ leads to an output-oriented, radial TI measurement, or the inverse of Farrell's output efficiency $\phi_Y(\boldsymbol{x}_0, \boldsymbol{y}_0) = \max\{\phi : \phi\boldsymbol{y}_0 \in Y(\boldsymbol{x}_0)\} \geq 1$ for the output set $Y(\boldsymbol{x})$ associated with T.[2]

Profit function $\pi_T(\boldsymbol{w}, \boldsymbol{p})$ attains the highest production value in technology T for given input-output prices $(\boldsymbol{w}, \boldsymbol{p}) \in \mathbb{R}_+^{L+M}$;

$$\begin{aligned}\pi_T(\boldsymbol{w}, \boldsymbol{p}) &= \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x} : \ (\boldsymbol{x}, \boldsymbol{y}) \in T\} \\ &= \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x} + D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)(\boldsymbol{p}\boldsymbol{g}_y + \boldsymbol{w}\boldsymbol{g}_x)\}\end{aligned} \tag{2}$$

where the second expression follows from the definition of directional distance function $(\boldsymbol{x} - D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)\boldsymbol{g}_x, \boldsymbol{y} + D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)\boldsymbol{g}_y) \in T$. The duality between the profit function and the directional distance function (Chambers, Chung, and Fre, 1998) yields;

$$D_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = \min_{\boldsymbol{w}',\boldsymbol{p}'}\left\{\frac{\pi_T(\boldsymbol{p}', \boldsymbol{w}') - (\boldsymbol{p}'\boldsymbol{y} - \boldsymbol{w}'\boldsymbol{x})}{\boldsymbol{p}'\boldsymbol{g}_y + \boldsymbol{w}'\boldsymbol{g}_x}\right\}, \tag{3}$$

which shows that in the set of supporting hyperplanes for technology T, TI is obtained at the shadow values that evaluate the decision most favorably. For given market prices $(\boldsymbol{w}, \boldsymbol{p})$, profit inefficiency (PI) is defined as

$$D_{PF}(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = \frac{\pi_T(\boldsymbol{p}, \boldsymbol{w}) - (\boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x})}{\boldsymbol{p}\boldsymbol{g}_y + \boldsymbol{w}\boldsymbol{g}_x} \tag{4}$$

where the subscript PF refers to profit-function(PF) technology $T_{PF}(\boldsymbol{w}, \boldsymbol{p})$, which is cast as a hypothetical technology that envelopes T under linear techical substitutability and takes the form

[1] The directional distance function is the technology-counterpart to the shortage function of Luenberger (1994).
[2] $\theta_V(\boldsymbol{x}_0, \boldsymbol{y}_0) = 1/(1 - D_T(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{x}_0, \boldsymbol{0}))$ and $\phi_Y(\boldsymbol{x}_0, \boldsymbol{y}_0) = 1/(1 - D_T(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{0}, \boldsymbol{y}_0))$.

of a half-space bounded by the profit function, or $T_{PF}(\boldsymbol{w},\boldsymbol{p}) = \{(\boldsymbol{x},\boldsymbol{y}) : \boldsymbol{py} - \boldsymbol{wx} \leq \pi_T(\boldsymbol{w},\boldsymbol{p})\}$. Strictly speaking, the PI in (4) and the PF technology should be represented conditionally on $(\boldsymbol{w},\boldsymbol{p})$, yet this notation is omitted by assuming fixed market prices. At direction $(-\boldsymbol{g}_x,\boldsymbol{g}_y) = (\boldsymbol{0},\boldsymbol{g}_y)$ or $(-\boldsymbol{g}_x,\boldsymbol{g}_y) = (-\boldsymbol{g}_x,\boldsymbol{0})$, PF frontier reduces to a revenue or cost frontier for the associated revenue or cost function respectively. According to additive decomposition $PI = TI + AI$,[3] the difference between the PI in (4) and the TI in (3) defines allocative inefficiency (AI);

$$AI(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) = D_{PF}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) - D_T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y). \tag{5}$$

Any input-output bundle on the frontier of T_{PF} is allocatively-efficient at prices $(\boldsymbol{w},\boldsymbol{p})$ and attains the maximum profit at $\pi_T(\boldsymbol{w},\boldsymbol{p})$.

The assumption of constant returns to scale (CRS) considers a hypothetical technology that envelops T under the linear homogeneity of input-output relationships, or $T_{CRS} = \cup_{\lambda\in\mathbb{R}_+}\lambda T$.[4] Denote the profit function associated with T_{CRS} by; $\pi_{CRS}(\boldsymbol{w},\boldsymbol{p}) = \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{py} - \boldsymbol{wx} : (\boldsymbol{x},\boldsymbol{y}) \in T_{CRS}\}$ which equals 0 if $\pi_T(\boldsymbol{w},\boldsymbol{p}) \leq 0$ and ∞ if $\pi_T(\boldsymbol{w},\boldsymbol{p}) > 0$.[5] Assuming $\pi_{CRS}(\boldsymbol{w},\boldsymbol{p}) = 0$ (e.g., equilibrium outcome under perfect competition with free entry and exit), the corresponding pseudo-TI measurement under CRS, say TI(CRS), is;

$$D_{CRS}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) = \min_{\boldsymbol{p}',\boldsymbol{w}'}\left\{\frac{\pi_{CRS}(\boldsymbol{p}',\boldsymbol{w}') - (\boldsymbol{p}'\boldsymbol{y} - \boldsymbol{w}'\boldsymbol{x})}{\boldsymbol{p}'\boldsymbol{g}_y + \boldsymbol{w}'\boldsymbol{g}_x}\right\}, \tag{6}$$

which is positive and bounded. According to additive decomposition $TI(CRS) = TI + SI$, the difference between the TI(CRS) in (6) and the TI in (3) defines scale inefficiency (SI);

$$SI(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) = D_{CRS}(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y) - D_T(\boldsymbol{x},\boldsymbol{y};\boldsymbol{g}_x,\boldsymbol{g}_y). \tag{7}$$

Any decision on the frontier of T_{CRS} is scale-efficient.

2.2 Weighted DEA (WDEA) Approach

Turning to empirical efficiency measurements, weighted DEA (WDEA) approach is presented in below for input-output bundles $\{(\boldsymbol{x}_i,\boldsymbol{y}_i)\}_{i\in\mathbb{I}}$ with observations indexed by $\mathbb{I} = \{1...N\}$.

[3]The multiplicative decomposition in the form of $PI = TI * AI$ is referred to as Nerlovian profit inefficiency measurement as it first appeared in Nerlove (1965). Its PI measure is given as $\pi(\boldsymbol{w},\boldsymbol{p})/(\boldsymbol{py} - \boldsymbol{wx})$.

[4] For input-output relationships specified in physical quantiles, linear-homogeneity is most suitably defined with respect to the origin. For those specified in monetary variables or qualitative indices, one may postulate a shifted CRS (SCRS) technology under a pseudo CRS-assumption around arbitrary point $(\boldsymbol{x}^o,\boldsymbol{y}^o)$ instead of the origin, or $T_{SCRS} = (\boldsymbol{x}^o,\boldsymbol{y}^o) + \cup_{\lambda\in\mathbb{R}_+}\lambda(T - (\boldsymbol{x}^o,\boldsymbol{y}^o))$. The associated profit function is $\pi_{SCRS}(\boldsymbol{w},\boldsymbol{p}) = \boldsymbol{py}^o - \boldsymbol{wx}^o + \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{py} - \boldsymbol{wx} : (\boldsymbol{x},\boldsymbol{y}) \in T_{SCRS}\}$ which equals $\boldsymbol{py}^o - \boldsymbol{wx}^o$ if $\pi_T(\boldsymbol{w},\boldsymbol{p}) \leq \boldsymbol{py}^o - \boldsymbol{wx}^o$ or ∞ if $\pi_T(\boldsymbol{w},\boldsymbol{p}) > \boldsymbol{py}^o - \boldsymbol{wx}^o$.

[5]Under the CRS assumption, zero-profit is always feasible by the feasible inaction $(\boldsymbol{0},\boldsymbol{0}) \in T_{CRS}$ at $\lambda = 0$.

The DEA approximations under VRS and CRS are respectively the free-disposal convex hull of data points (i.e., all convex combinations of data points and the points implied by free-disposabiliity) and the free-disposal conical hull of data points (i.e., every point in $\widehat{T}_{VRS}$ and any scaler multiple of it), or

$$\widehat{T}_{VRS} = \{(\boldsymbol{x}', \boldsymbol{y}') : \ \sum_{j\in\mathbb{I}} \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}', \ , \sum_{j\in\mathbb{I}} \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x}', \ \sum_{j\in\mathbb{I}} \lambda_j = 1, \ \boldsymbol{\lambda} \in \mathbb{R}^N_+\}, \tag{8}$$

$$\widehat{T}_{CRS} = \{(\boldsymbol{x}', \boldsymbol{y}') : \ \sum_{j\in\mathbb{I}} \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y}', \ , \sum_{j\in\mathbb{I}} \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x}', \ \boldsymbol{\lambda} \in \mathbb{R}^N_+\}. \tag{9}$$

$\widehat{T}_{VRS}$ corresponds to the smallest producible set satisfying assumptions A.1-A.3, while $\widehat{T}_{CRS}$ envelops $\widehat{T}_{VRS}$ under linear homogeneity. The estimates for TI and TI(CRS) by the directional distance function in (1) are $\widehat{D}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b : (\boldsymbol{x}_0 - b\boldsymbol{g}_x, \boldsymbol{y}_0 + b\boldsymbol{g}_y) \in \widehat{T}_{VRS}\}$ and $\widehat{D}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y) = \max\{b : (\boldsymbol{x}_0 - b\boldsymbol{g}_x, \boldsymbol{y}_0 + b\boldsymbol{g}_y) \in \widehat{T}_{CRS}\}$ respectively. The dual problem for $\widehat{D}_{VRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y)$, corresponding to the dual representation in (3), is

$$\min\{\rho \in \mathbb{R} : \ \forall j \in \mathbb{I}, \ \boldsymbol{p}\boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j \leq \boldsymbol{p}\boldsymbol{y}_0 - \boldsymbol{w}\boldsymbol{x}_0 + \rho, \ \boldsymbol{p}\boldsymbol{g}_y + \boldsymbol{w}\boldsymbol{g}_x = 1, \ \boldsymbol{p} \in \mathbb{R}^M_+, \ \boldsymbol{w} \in \mathbb{R}^L_+\}, \tag{10}$$

which minimizes TI-parameter ρ subject to the optimality of shadow value $\boldsymbol{p}\boldsymbol{y}_0 - \boldsymbol{w}\boldsymbol{x}_0 + \rho$ for decision $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, given the feasibility constraints under $\widehat{T}_{VRS}$ and price normalization $\boldsymbol{p}\boldsymbol{g}_y + \boldsymbol{w}\boldsymbol{g}_x = 1$. The dual estimation for $\widehat{D}_{CRS}(\boldsymbol{x}_0, \boldsymbol{y}_0; \boldsymbol{g}_x, \boldsymbol{g}_y)$, corresponding to (6), is obtained by imposing additional constraint $\boldsymbol{p}\boldsymbol{y}_0 - \boldsymbol{w}\boldsymbol{x}_0 + \rho = 0$ in problem (10), as implied by condition $\pi_{CRS}(\boldsymbol{p}, \boldsymbol{w}) = 0$.

Profit-function (PF) technology is estimated as the half space bounded by $\widehat{\pi}_{VRS}(\boldsymbol{w}, \boldsymbol{p})$;

$$\begin{aligned} \widehat{T}_{PF} &= \{(\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x} \leq \widehat{\pi}_{VRS}(\boldsymbol{w}, \boldsymbol{p})\} \quad \text{where} \\ \widehat{\pi}_{VRS}(\boldsymbol{w}, \boldsymbol{p}) &= \max_{\boldsymbol{x},\boldsymbol{y}}\{\boldsymbol{p}\boldsymbol{y} - \boldsymbol{w}\boldsymbol{x} : \ (\boldsymbol{x}, \boldsymbol{y}) \in \widehat{T}_{VRS}\} = \max_{j\in\mathbb{I}}\{\boldsymbol{p}\boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j\}. \end{aligned} \tag{11}$$

The conventional measures of technical, allocative, and scale inefficiencies ($\widehat{TI}_{VRS}$, $\widehat{AI}_{VRS}$, and $\widehat{SI}_{VRS}$) are estimated as distances (3), (5), and (7) respectively using frontier approximations (8), (9), and (11). The standard practice is to utilize technology approximation $\widehat{T}_{VRS}$, from which TI, AI, and SI are measured. While these estimates are consistent, more efficient estimation can be devised under a simultaneous estimation of the technology and inefficiency concepts.

To this end, the current study proposes a weighted DEA (WDEA) approach to integrating the concepts of TI, AI, and SI into a technology approximation. Consider WDEA technology

$\widehat{T}_{W(\alpha,\beta)}$ defined as the weighted average of $\widehat{T}_{VRS}$, $\widehat{T}_{PF}$, and $\widehat{T}_{CRS}$ for given weights $\{1-\alpha-\beta,$ $\alpha,$ $\beta\}$ respectively;[6]

$$\begin{aligned}\widehat{T}_{W(\alpha,\beta)} &\equiv (1-\alpha-\beta)\widehat{T}_{VRS} + \alpha\widehat{T}_{PF} + \beta\widehat{T}_{CRS} \\ &= \widehat{T}_{VRS} + \alpha(\widehat{T}_{PF} - \widehat{T}_{VRS}) + \beta(\widehat{T}_{CRS} - \widehat{T}_{VRS}),\end{aligned} \tag{12}$$

which expands the conventional producible set $\widehat{T}_{VRS}$ by α-portion of the input-output space conventionally regarded as AI and β-portion of the space conventionally regarded as SI.[7] Arbitrary weights α and β respectively generalize the extents of linear substitution and linear homogeneity assumptions in DEA. Consequently, $\widehat{T}_{W(\alpha,\beta)}$ includes the conventional DEA frontiers of PF, CRS, and VRS as special cases; $\widehat{T}_{W(0,0)} = \widehat{T}_{VRS}$, $\widehat{T}_{W(1,0)} = \widehat{T}_{PF}$, and $\widehat{T}_{W(0,1)} = \widehat{T}_{CRS}$. If weights α and β fall outside of range $[0,1]$, the technical feasibility can be defined as $\widehat{T}^{*}_{W(\alpha,\beta)} \equiv \widehat{T}_{W(\alpha,\beta)} \cup \widehat{T}_{VRS}$, so that the WDEA approximation of a technology is bounded from below by $\widehat{T}_{VRS}$.

For decision $(\boldsymbol{x}_0, \boldsymbol{y}_0)$, let the TI measured under WDEA technology $\widehat{T}_{W(\alpha,\beta)}$ be

$$\begin{aligned}\widehat{D}_{W(\alpha,\beta)}(\boldsymbol{x}_0,\boldsymbol{y}_0) &= (1-\alpha-\beta)\widehat{D}_{VRS}(\boldsymbol{x}_0,\boldsymbol{y}_0) + \alpha\widehat{D}_{PF}(\boldsymbol{x}_0,\boldsymbol{y}_0) + \beta\widehat{D}_{CRS}(\boldsymbol{x}_0,\boldsymbol{y}_0) \\ &= \widehat{TI}_{VRS}(\boldsymbol{x}_0,\boldsymbol{y}_0) + \alpha\widehat{AI}_{VRS}(\boldsymbol{x}_0,\boldsymbol{y}_0) + \beta\widehat{SI}_{VRS}(\boldsymbol{x}_0,\boldsymbol{y}_0),\end{aligned} \tag{13}$$

and let the associated AI and SI measures be

$$\widehat{AI}_{W(\alpha,\beta)}(\boldsymbol{x}_0,\boldsymbol{y}_0) = \widehat{D}_{PF,W}(\boldsymbol{x}_0,\boldsymbol{y}_0) - \widehat{D}_{W(\alpha,\beta)}(\boldsymbol{x}_0,\boldsymbol{y}_0), \tag{14}$$

$$\widehat{SI}_{W(\alpha,\beta)}(\boldsymbol{x}_0,\boldsymbol{y}_0) = \widehat{D}_{CRS,W}(\boldsymbol{x}_0,\boldsymbol{y}_0) - \widehat{D}_{W(\alpha,\beta)}(\boldsymbol{x}_0,\boldsymbol{y}_0) \tag{15}$$

where $\widehat{D}_{CRS,W}(\boldsymbol{x}_0,\boldsymbol{y}_0)$ and $\widehat{D}_{PF,W}(\boldsymbol{x}_0)$ are obtained by replacing $\widehat{T}_{VRS}$ in (9) and (11) with $\widehat{T}_{W(\alpha,\beta)}$ respectively. Note that at $\beta = 0$, the new AI measurement reduces to;

$$\widehat{AI}_{W(\alpha,0)}(\boldsymbol{x}_i,\boldsymbol{y}_i) = (1-\alpha)(\widehat{D}_{PF}(\boldsymbol{x}_i,\boldsymbol{y}_i) - \widehat{D}_{VRS}(\boldsymbol{x}_i,\boldsymbol{y}_i)) = (1-\alpha)\widehat{AI}_{VRS}(\boldsymbol{x}_i,\boldsymbol{y}_i). \tag{16}$$

Similarly, at $\alpha = 0$, the new SI measurement reduces to;

$$\widehat{SI}_{W(0,\beta)}(\boldsymbol{x}_i,\boldsymbol{y}_i) = (1-\beta)(\widehat{D}_{CRS}(\boldsymbol{x}_i,\boldsymbol{y}_i) - \widehat{D}_{VRS}(\boldsymbol{x}_i,\boldsymbol{y}_i)) = (1-\beta)\widehat{SI}_{VRS}(\boldsymbol{x}_i,\boldsymbol{y}_i). \tag{17}$$

The next subsection considers an optimal weight selection for α and β. In below, simplified

[6] While notation $(\boldsymbol{w},\boldsymbol{p})$ is omitted, $\widehat{T}_{W(\alpha,\beta)}$ clearly depends on the market prices through the profit frontier along $\widehat{T}_{PF}$.

[7] Alternatively, one can define $\widehat{T}_{W(\alpha,\beta)}$ with a convex combination of arbitrary direction $(-\tilde{\boldsymbol{g}}_x, \tilde{\boldsymbol{g}}_y)$ instead of the current, radial orientation.

notations are used for distance function $D_{T,i} \equiv D_T(\boldsymbol{x}_i, \boldsymbol{y}_i)$ and inefficiencies $TI_{T,i} \equiv TI_T(\boldsymbol{x}_i, \boldsymbol{y}_i)$, $AI_{T,i} \equiv AI_T(\boldsymbol{x}_i, \boldsymbol{y}_i)$, and $SI_{T,i} \equiv SI_T(\boldsymbol{x}_i, \boldsymbol{y}_i)$.

2.3 Weight Selection

Consider the following weight selection mechanism that proceeds in two steps. The first step makes some initial estimate $\widehat{D}_{T,i}$ at the observation level and the second step predicts sample-level relationships between this estimate $\widehat{D}_{T,i}$ and the conventional measures of TI, AI, and SI. Namely, the second step estimates optimal weights by minimizing least square errors of the form;

$$\{\hat{\alpha}, \hat{\beta}\} = \underset{\alpha,\beta}{\operatorname{argmin}} \left\{ \frac{1}{N} \sum_{i \in \mathbb{I}} \left(\widehat{D}_{T,i} - (\widehat{TI}_{VRS,i} + \alpha \widehat{AI}_{VRS,i} + \beta \widehat{SI}_{VRS,i}) \right)^2 \right\}, \tag{18}$$

which is the moment condition implied by equation (13) when $\widehat{D}_{W(\alpha,\beta)i}$ is substituted with some estimate $\widehat{D}_{T,i}$ from the first step. The remainder of this section describes the first-step estimation for $\widehat{D}_{T,i}$, discusses some properties of this weight selection, and provides a simple, illustrative example.

The conceptual underpinning for the first-step estimate $\widehat{D}_{T,i}$ draws on the subsample-bootstrap estimator proposed by Kneip, Simar, and Wilson (2008). The authors showed that for a convex technology, the behavior of the VRS estimator can be analyzed through the relative frequency for observations to be located in a small neighborhood around the true frontier. Assuming a uniform density in the neighborhood, they derived an asymptotic distribution of this estimator. Given the equivalence between the asymptotic properties of (additive) directional distance functions and those of (multiplicative) radial inefficiency measures (Simar, Vanhems, and Wilson, 2012), the $1-a$ confidence interval for $\widehat{D}_{VRS,i}$ can be written as;

$$1 - a = Pr(C_a \leq \widehat{D}_{VRS,i} - D_T \leq C_b) \approx Pr(C_a \leq \widehat{D}^*_{VRS,i} - \widehat{D}_{VRS} \leq C_b) \tag{19}$$

where C_a and C_b represent lower and upper critical values for the deviation, and $\widehat{D}^*_{VRS,i}$ is a bootstrap VRS estimator using $K(< N)$ observations sampled without replacement.[8] The critical values are substituted with estimates $\widehat{C}_a = \psi_{a/2,K}$ and $\widehat{C}_b = \psi_{1-a/2,K}$ where $\psi_{x,K} \leq 0$ denotes the x-quantile of the bootstrap distribution $\{K^{2/(L+M+1)}(\widehat{D}^{*,b}_{VRS,i} - \widehat{D}_{VRS,i})\}_{b=1}^B$ from B bootstrap replications. The intuition behind the subsample-bootstrapping is that the distribution of the difference $\widehat{D}_{VRS,i} - D_T$ between the VRS estimator (in the sample) and the true value (in the universe) can be predicted from the distribution of the difference $\widehat{D}^*_{VRS,i} - \widehat{D}_{VRS}$ between the

[8] Another approach would be the smooth-bootstrap method of Kneip, Simar, and Wilson (2011).

bootstrap-VRS estimator (in a subsample) and the VRS estimator (in the full sample), given the adjustments for the different rates of convergence under different sample sizes. Then, the confidence interval in (19) can be estimated as

$$[\widehat{D}_{VRS,i} - N^{-2/(L+M+1)}\psi_{1-a/2,K}, \quad \widehat{D}_{VRS,i} - N^{-2/(L+M+1)}\psi_{a/2,K}], \tag{20}$$

which reflects the accuracy of local VRS estimator $\widehat{D}_{VRS,i}$, predicted from the local sample density in the neighborhood. Let the mean of this confidence interval be referred to as mean bootstrap (MB) estimator (e.g., Simar, Vanhems, and Wilson, 2012), which makes upward adjustments to conventional TI estimate $\widehat{D}_{VRS,i}$;[9]

$$\widehat{D}_{MB,i} = \widehat{D}_{VRS,i} - \left(\frac{K}{N}\right)^{2/(L+M+1)} \frac{1}{B}\sum_{b=1}^{B}(\widehat{D}^{*,b}_{VRS,i} - \widehat{D}_{VRS,i}) \tag{21}$$

where $\widehat{D}^{*,b}_{VRS,i} - \widehat{D}_{VRS,i} \leq 0$. In effect, MB estimator tends to assume a larger producible set than $\widehat{T}_{VRS}$ everywhere along the frontier.

Two modifications to $\widehat{D}_{MB,i}$ are made before arriving at the proposed estimate for $\widehat{D}_{T,i}$. One modification is to correct for systematic bias in $\widehat{D}_{MB,i}$ with respect to the scales of operations. The bias arises from the potential inapplicability of TI measurements under random subsamples (, which may not contain reference observations for sufficiently small- or large-scale operations) and is systematically related to the direction of TI measurement; small-scale decisions cannot be assessed for output-oriented TI if comparably-small scale input-decisions are absent in the subsample, and similarly large-scale decisions cannot be evaluated for input-oriented TI if comparably-large scale output-decisions are absent. Using only the estimable cases of bootstrap-TI measurements would underestimate the bias-corrections in (21) for these decisions. By simultaneously employing input- and output-oriented MB estimators, denoted by $\widehat{D}^{I}_{MB,i}$ and $\widehat{D}^{O}_{MB,i}$ respectively, scale-neutral MB technology can be specified as;

$$\begin{aligned}\widehat{T}^{N}_{MB} = \{(\boldsymbol{x}', \boldsymbol{y}') : \ & \sum_{j\in I}\lambda_j(y_j + \widehat{D}^{O}_{MB,i}) + \sum_{j\in I}\eta_j \boldsymbol{y}_j \geq \boldsymbol{y}', \\ & \sum_{j\in I}\lambda_j \boldsymbol{x}_j + \sum_{j\in I}\eta_j(\boldsymbol{x}_j - \widehat{D}^{I}_{MB,i}) \leq \boldsymbol{x}', \ \sum_{j\in I}\lambda_j + \sum_{j\in I}\eta_j = 1, \ \boldsymbol{\lambda}, \boldsymbol{\eta} \in \mathbb{R}^{N}_{+}\},\end{aligned} \tag{22}$$

which yields associated estimator $\widehat{D}^{N}_{MB,i} = \max\{b : (\boldsymbol{x}_i - b\boldsymbol{g}_x, \boldsymbol{y}_i + b\boldsymbol{g}_y) \in \widehat{T}^{N}_{MB}\}$ for direction $(\boldsymbol{g}_x, \boldsymbol{g}_y)$.

[9]The mean can be replaced with the median or mode of distribution $\{K^{2/(L+M+1)}(\widehat{D}^{*,b}_{VRS,} - \widehat{D}_{VRS,i})\}^{B}_{b=1}$. Simulation study may be helpful to investigate these alternative estimators.

The other modification is to reduce the magnitude of upward adjustments in (21), such that some observed decisions can be regarded fully-technically efficient under the implied technology. This prevents the model from postulating a strictly larger technical feasibility than $\widehat{T}_{VRS}$. For constant $\bar{c} = E[\widehat{D}^N_{MB,i} - \widehat{D}_{VRS,i} - \widehat{D}_{VRS,i} | \widehat{D}_{VRS,i} = 0]$, consider shifted-mean bootstrap (SMB) estimator $\bar{c} = E[\widehat{D}^N_{MB,i} - \widehat{D}_{VRS,i} | \widehat{D}_{VRS,i} = 0]$. Thus, constant $\bar{c}$, the mean difference between the above MB and VRS estimators conditionally on being technically-efficient under the VRS frontier, is used to shift back and normalize the upward adjustment $\widehat{D}^N_{MB,i} - \widehat{D}_{VRS,i}$. The lower bound for TI by $\widehat{D}_{VRS,i}$ is added to ensure that the associated technical feasibility is bounded from below by $\widehat{T}_{VRS}$. It may be noted that the magnitude of constant $\bar{c}$ directly affects the mean TI under SMB and hence the mean TI under WDEA.[10]

Hence, the proposed weight selection first estimates $\widehat{D}_{T,i}$ by $\widehat{D}_{SMB,i}$ and then weights $\hat{\alpha}$ and $\hat{\beta}$ by equation (18). These weights represent the sample-level relationships between locally-derived adjustments $\widehat{D}_{SMB,i} - \widehat{D}_{VRS,i}$ (i.e., predicted bias corrections for the conventional TI measure $\widehat{D}_{VRS,i}$) and the conventional measures of AI and SI. By accounting for the sample correlations among these inefficiency concepts, WDEA technology $\widehat{T}_{W(\hat{\alpha},\hat{\beta})}$ systematically extends conventional technology approximation $\widehat{T}_{VRS}$, or the smallest feasible set meeting assumptions A1-A3.

Some properties of the WDEA estimator are noted with respect to the following relationships between unobserved $D_{T,i}$ and its estimates by VRS, SMB, and WDEA;

$$\begin{aligned}
VRS: &\quad D_{T,i} = \widehat{TI}_{VRS,i} + \varepsilon_{VRS,i}, \quad \varepsilon_{VRS,i} > 0 \\
SMB: &\quad D_{T,i} = \widehat{D}_{SMB,i} + \varepsilon_{SMB,i}, \quad E[\varepsilon_{SMB,i}] = 0 \\
WDEA: &\quad D_{T,i} = \widehat{TI}_{VRS,i} + \alpha \widehat{AI}_{VRS,i} + \beta \widehat{SI}_{VRS,i} + \varepsilon_{W(\alpha,\beta),i}, \quad E[\varepsilon_{W(\alpha,\beta),i}] = 0
\end{aligned} \tag{23}$$

where $\varepsilon_{VRS,i}$, $\varepsilon_{SMB,i}$, and $\varepsilon_{W(\alpha,\beta),i}$ are residual terms that close these identities. In the first equation, the well-known one-sided bias of the VRS estimator (i.e. $\varepsilon_{VRS,i} > 0$) implies mean-inconsistency $E[D_{T,i} - \widehat{TI}_{VRS,i}] = E[\varepsilon_{VRS,i}] > 0$, while it is asymptotically consistent, or $E[D_{T,i} - \widehat{TI}_{VRS,i}] \to 0$ for a sufficiently large sample (Banker, Gadh, and Gorr, 1993). In the second equation, the SMB estimator is assumed to be consistent, so that $E[D_{T,i} - \widehat{D}_{SMB,i}] = E[\varepsilon_{SMB,i}] = 0$. Given this assumption, combining the second and the third equations to eliminate $D_{T,i}$ and using $\hat{\alpha}$ and $\hat{\beta}$ in (18) yield a consistent WDEA estimator;

Remark 1. *In* (18) *and* (23)*, if* $E[\varepsilon_{SMB,i} | \widehat{D}_{SMB,i}, \widehat{TI}_{VRS,i}, \widehat{AI}_{VRS,i}, \widehat{SI}_{VRS,i}] = 0$*, then the*

[10] An alternative for $\bar{c}$ is to use the minimum of the difference $\widehat{D}^N_{MB,i} - \widehat{D}_{VRS,i}$ instead of the mean. Yet, given the relative inaccuracy in predicting $\widehat{T}^N_{MB}$, the use of the mean difference appears more reliable.

WDEA estimator is consistent, or $E[\varepsilon_{W(\hat{\alpha},\hat{\beta}),i}] = 0$.

Turning to estimation efficiency, simple comparisons are noted;

Remark 2. *In* (23)*, if* $E[\varepsilon_{SMB,i}|\widehat{D}_{SMB,i}, \widehat{TI}_{VRS,i}] = 0$*, then the SMB estimator is more efficient than the VRS estimator in that* $E[(\varepsilon_{SMB,i})^2] \leq E[(\varepsilon_{VRS,i})^2]$ *where* $\varepsilon_{VRS,i} = (\widehat{D}_{SMB,i} - \widehat{TI}_{VRS,i}) + \varepsilon_{SMB,i}$.

Remark 3. *In* (18) *and* (23)*, if* $E[\varepsilon_{W(\hat{\alpha},\hat{\beta}),i}|\widehat{AI}_{VRS,i}, \widehat{SI}_{VRS,i}] = 0$ *and* $\hat{\alpha}$, $\hat{\beta} \geq 0$*, then the WDEA estimator is more efficient than the VRS estimator in that* $E[(\varepsilon_{W(\hat{\alpha},\hat{\beta}),i})^2] \leq E[(\varepsilon_{VRS,i})^2]$ *where* $\varepsilon_{VRS,i} = \alpha\widehat{AI}_{VRS,i} + \beta\widehat{SI}_{VRS,i} + \varepsilon_{W(\hat{\alpha},\hat{\beta}),i}$.

Remark 4. *In* (18) *and* (23)*, if* $E[\varepsilon_{SMB,i}|\widehat{D}_{SMB,i}, \widehat{TI}_{VRS,i}, \widehat{AI}_{VRS,i}, \widehat{SI}_{VRS,i}] = 0$ *and* $\hat{\alpha}$, $\hat{\beta} \leq 0$*, then the SMB estimator is more efficient than the WDEA estimator in that* $E[(\varepsilon_{SMB,i})^2] \leq E[(\varepsilon_{W(\hat{\alpha},\hat{\beta}),i})^2]$ *where* $\varepsilon_{SMB,i} = \alpha\widehat{AI}_{VRS,i} + \beta\widehat{SI}_{VRS,i} + (\widehat{TI}_{VRS,i} - \widehat{D}_{SMB,i}) + \varepsilon_{W(\alpha,\beta),i}$.

Remark 2 follows from $\widehat{D}_{SMB,i} - \widehat{TI}_{VRS,i} \geq 0$. Remark 3 similarly follows under α, $\beta \geq 0$. Remark 4 states that under α, $\beta \leq 0$, incorporating AI and SI into a technology estimation would be counterproductive. Meanwhile, there seems no simple condition that ensures higher efficiency of the WDEA estimator than the SMB counterpart.

The following example illustrates a process of constructing (simplified versions of) SMB and WDEA estimators. Consider a case of one-input, one-output production with a sample of 6 observations (x_i, y_i), $i = 1, .., 6$. Figure 3 depicts relative geometric locations of these observations, labeled A_1-A_6. Points A_1,A_2, and A_3 are technically-efficient under VRS but only point A_2 is technically-efficient under CRS. Points A_4, A_5, and A_6 are all technically-inefficient under VRS and are less efficient versions of points A_1,A_2, and A_3 respectively, such that $x_1 = x_4 < x_2 = x_5 < x_3 = x_6$ and $y_1 > y_4$, $y_2 > y_5$, $y_3 > y_6$. For the ease of illustration, consider the SMB estimator based solely on the output-oriented TI by $\widehat{D}^O_{MB,i}$ (without using the output-oriented TI by $\widehat{D}^I_{MB,i}$) and single WDEA weight $\hat{\beta} > 0$ (with $\hat{\alpha} = 0$). In relation to the total number of observations $N = 6$, the number of subsample is set at $K = 1$ for simplicity. At $K = 1$, bootstrap VRS frontier reduced to a free disposable hull (FDH), so that $\widehat{D}^O_{MB,i}$ can be described as the difference in outputs of two decisions. With a sufficient number of bootstrap replications $b = 1, .., B$, each data point is drawn at probability $p_i = 1/6$, and the mean bootstrap estimate $\widehat{D}_{MB,i}$ $(1/B)\sum_b(\widehat{D}^{*,b}_{VRS,i} - \widehat{D}_{VRS,i})$ converges to its expected value. Then, without loss of generality, by treating observation index i interchangeably with bootstrap index $b = 1, .., 6$,

the MB estimator for point A_1 can be described as

$$\begin{aligned}\widehat{D}_{MB,1} &= \widehat{D}_{VRS,1} - (K/N)^{2/(L+M+1)}[\tilde{p}_1(\widehat{D}^{*,b=1}_{VRS,1} - \widehat{D}_{VRS,1}) + \tilde{p}_4(\widehat{D}^{*,b=4}_{VRS,1} - \widehat{D}_{VRS,1})] \\ &= 0 - C_0[\frac{1}{2}(y_1 - y_1 - 0) + \frac{1}{2}(y_4 - y_1 - 0)] = C_0\, E_b[y_1 - y_b\,|b = 1, 4] \qquad (24)\end{aligned}$$

where $C_0 = (K/N)^{2/L+M+1}$ is a constant ($(1/6)^{2/3}$ in this example), and $\tilde{p}_1 = \tilde{p}_4 = 1/2$ is a pseudo-probability defined conditionally on the feasible TI estimation under bootstrap replications $b = 1, 4$. [11] Similarly, those for points A_2 to A_6 are $\widehat{D}_{MB,2} = C_0\, E_b[y_2 - y_b\,|b = 1, 2, 4, 5]$, $\widehat{D}_{MB,3} = C_0 E_b[y_3 - y_b]$, $\widehat{D}_{MB,4} = (y_1 - y_4) + C_0 E_b[y_1 - y_b|b = 1, 4]$, $\widehat{D}_{MB,5} = (y_2 - y_5) + C_0 E_b[y_2 - y_b|b = 1, 2, 4, 5]$, and $\widehat{D}_{MB,6} = (y_3 - y_6) + C_0 E_b[y_1 - y_b]$. Thus, the local frontier levels are adjusted by C_0 times expected bootstrap deviation $E_b[y_i - y_b]$. By setting $\bar{c} = (1/3)\sum_{j=1,2,3}[\widehat{D}_{MB,j} - \widehat{D}_{VRS,j}|\widehat{D}_{VRS,j} = 0]$ and $\widehat{D}_{SMB,i} = \max\{(\widehat{D}_{MB,j} - \widehat{D}_{VRS,j} - \bar{c})/(1+\bar{c}),\, \widehat{D}_{VRS,i}\}$ for $i = 1, .., 6$, optimal weight is estimated by $\hat{\beta} = Cov(\widehat{D}_{SMB,i} - \widehat{D}_{VRS,i},\, \widehat{D}_{CRS,i} - \widehat{D}_{VRS,i})/Var(\widehat{D}_{CRS,i} - \widehat{D}_{VRS,i}) = Cov(\widehat{D}_{SMB,i} - \widehat{TI}_{VRS,i}, \widehat{SI}_{VRS,i})/Var(\widehat{SI}_{VRS,i})$ where $Cov(.)$ and $Var(.)$ denote the covariance and variance operators respectively.

Figure 4 sketches the SMB estimates (at points C_1, C_2, and C_3) and WDEA estimates (at points D_1, D_2, and D_3) in the above example. The SMB estimator yields no adjustment at A_1, a small expansion of technical feasibility at A_2, and a large expansion at A_3, according to the local performance of VRS estimators assessed by the bootstrapping process. The WDEA estimator consolidates these local adjustments into systematic frontier expansions from the VRS frontier toward the CRS frontier at weight $\hat{\beta}$, resulting in moderate expansions of technical feasibility at A_1 and A_3 with no expansion at scale-efficient point A_2. Weight $\hat{\beta}$ depends on the covariance between SMB's adjustments to TI (i.e., distances C_1A_1, C_2A_2, and C_3A_3 in figure 4) and conventional SI measures (i.e., distances B_1A_1, B_2A_2, and B_3A_3 in figure 3).

Hence, the proposed two-step weight selection is summarized as follows. The subsample bootstrapping in the first step links the performance of the VRS estimator to its local probability density. This yields a presumably-consistent adjustment at the observation level, yet its estimation efficiency depends on the nature of the data and the choice of subsample size that specifies the level of "locality." [12] The second-step regression summarizes this local adjustment into the sample correlations among TI, AI, and SI, all of which producers strive to minimize.

[11] The number of comparable points for calculating $\widehat{D}_{MB,i}$ increases in subsample size, making the issue of non-estimable TI situations less important.

[12] The confidence interval in (21) is rather sensitive to the choice of K (Kneip, Simar, and Wilson, 2008). The current application follows Simar and Wilson (2011)'s subsample-size selection that minimizes the volatility of the estimator.

3 Application

3.1 Data and Efficiency Measurements

Proposed WDEA approach is used to examine technical efficiencies and producer-specific input shadow values for Maryland dairy operations during 1995-2009. The dataset, previously analyzed in Hanson et al. (2013), contains unbalanced panel entries of production inputs and outputs. Each operation is categorized as either a conventional confinement dairy or management-intensive grazing dairy. The producer-year level observations represent 314 confinement operation-years and 164 grazing operation-years.

Dairy production is modeled with seven inputs; herd size (cows), hired labor, crop, animal, machinery, crop acreage, pasture acreage, where the four items from "labor" to "machinery" are quasi-quantity inputs measured as the corresponding categorical expenses divided by observation-specific price indices.[13] Table 1 provides summary statistics of milk output and these inputs. In the sample, a typical confinement operation produces about twice as much milk as a typical grazing operation and utilizes a 40% bigger herd, 486% more labor, 160% more crop-production inputs, 93% more animal-care inputs, 76% more machinery, 119% more crop acreage, and 67% less pasture acreage. In the current application, the two dairy systems are separately analyzed for their efficiency measurements.

Table 2 reports a summary of input-oriented radial efficiency scores ranging from 0 to 1 with 1 being fully-efficient.[14] Listed items TE(VRS), SMB, and WDEA report technical efficiency estimates against a VRS technology, a shifted mean bootstrap (SMB) technology, and a weighted DEA (WDEA) technology respectively. Items SE(VRS) and AE(VRS) are the scale and allocative efficiency estimates under the VRS technology. The mean scores of TE, SE, and AE under VRS are 0.896, 0.965, and 0.787 for confinement and 0.931, 0.914, and 0.717 for grazers respectively. The rather high SE scores are likely explained by the limited range of operational scales in the sample, ranging up to 468 cows for confinement and 195 cows for grazers; the inclusion of large-scale operations with over 1000 cows would lower these SE scores. The relatively low AE scores, on the other hand, suggest that in a short run these input mixes, often linked to long-term assets, are unlikely to be optimally allocated with respect to market prices.

[13] For example, the aggregate crop-related expenses for observation i is calculated as $p_i^{crop} = (seed_i/crop_i)\, p^{seed} + (fertilizer_i/crop_i)\, p^{fertilizer} + (chem_i/crop_i)\, p^{chem}$ where $crop_i = seed_i + fertilizer_i + chem_i$ for its total categorical expense for seeds, fertilizer, and chemicals. Price indices are obtained from Agricultural Statistical Service of USDA.

[14] Appendix A describes the market price estimation based on Kuosmanen, Cherchye, and Sipilinen (2006) with some additional constraints. The estimated prices in table 4 are used for obtaining allocative efficiency (AE) and WDEA. The primarily interest is the annual rental rates of dairy cows for confinement and grazers, estimated at \$575/cow and \$464/cow respectively. Under the expected 1.5 and 2.5 remaining-years of economically-viable milking for confinement and grazing cows, the culling value of \$500, and a 5% interest rate, these rental rates imply the present values of $\$575 + (\$575/2 + \$500)/1.05 \approx \$1325$, $\$464 + \$464/1.05 + (\$464/2 + \$500)/1.05^2 = \$1570$ respectively.

The mean TE scores under SMB and WDEA are 0.819 and 0.826 for confinement and 0.877 and 0.883 for grazers respectively.[15] The optimal weights for WDEA are estimated at $(\alpha, \beta) =$ (0.349, -0.209) for confinement and (0.138, -0.020) for grazers. The negative estimate for β for confinement is explained by a positive correlation (0.283) between AI and SI, indicating that among confinement operations, 34.9% of the apparent allocative inefficiency and minus 20.9% of the apparent scale inefficiency under the VRS technology is attributed to the underestimation in technical inefficiency. The increased discriminatory power under WDEA leads to the decreased mean TE scores by 0.070 (i.e. 7.8%) and 0.048 (5.2%) for confinement and grazers respectively, compared to the VRS results. This is similar to the finding in Brissimis, Delis, and Tsionas (2010) that their SFA frontier estimation with incorporating AE reduced TE by approximately 9%. The same intuition holds in both approaches; accounting for AE increases a linear substitutability of the predicted technological frontier and tends to lower the predicted TE for observed decisions. In a parametric model, moment conditions on AE can be used to augment a frontier estimation through distributional assumptions. In WDEA, the weight for linear substitution can be used to account for the correlation between the conventional AE and TE measures under a VRS frontier. Both approaches incorporate the concept of AE to enhance estimation efficiency.

3.2 Producer-Specific Shadow Values

Following Chambers and Färe (2008), the willingness to accept (WTA) and willingness to pay (WTP) for a change in input mixes are inferred by tracing the curvature along an estimated technological frontier. In theory, at an equilibrium in a frictionless economy, the WTA must be equal to or higher than the corresponding market price, and the WTP must be equal to or lower than the market price. The non-conformity of estimated shadow values to these theoretical predictions can provide insights into the state of factor markets or non-technological constraints for producers.

Figures 5 and 6 present the distributions of estimated WTA and WTP for confinement, plotted against the sample-proportion scaled from 0 to 1, and figures 7 and 8 those for grazers.[16] The WTA and WTP estimates vary substantially across observations. Under the VRS technology, the mean WTP for cow (per year rental rate) is \$652 for confinement and \$344 for grazers, and the mean WTA \$4082 and \$3562 respectively. Similarly, under the WDEA technology, the mean

[15]For SMB, the optimum sample sizes of input-oriented and output-oriented TE were found 157 and 267 for confinement and 121 and 129 for grazers. These values were searched from 10 equally-spaced values within the 40-90% (i.e. 40%,45%,...90%) of their sample sizes.

[16]The bottom figures in tables 6 and 8 contain the distributions of the estimated WTA and WTP for the bundle of a dairy cow and a crop acre. The WTA for the bundle is higher than the sum of WTA for individual inputs, and the WTP for the bundle is lower than the sum of WTP for individual inputs by the superadditivity of the directional derivative.

WTP for cow is \$323 for confinement and \$917 for grazers, and the mean WTA \$3348 and \$3575 respectively. Since the distributions of these shadow values are fat-tailed and contain extreme values, the averages are not be very informative. The distributions are alternatively are studied in relation to their market rates in below.

Somewhat surprisingly, no systematic patterns are discerned in the difference between the VRS and WDEA results for WTA and WTP estimates. This may strike some readers odd since the linear substitutability (as well as linear scalability) of a predicted technology would be systematically higher under WDEA than under VRS. In figures 1 and 2, for example, one may expect that WDEA yield a narrower range for the ratio of admissible shadow values, smaller gaps between WTA and WTP (e.g., the gap is infinite along a Leontief frontier and zero along a linear frontier), and generally smaller deviations of WTA and WTP from the market rates, compared to the VRS estimates. However, the relationships for the shadow values under VRS and WDEA frontiers in a simple two-dimensional diagram (e.g., see Appendix B) are generally inapplicable for higher dimensional input-output decisions.

The over-utilization and under-utilization of inputs are studied through the existence of unmet supply or demand for given market rate w_l^M for each input l, or latent marginal supply or demand (LMS or LMD) proportions defined as the sample proportions of observations $i \in \mathbb{I}$ satisfying $[WTA_{i,l} \leq w_l^M]$ or $[WTP_{i,l} \geq w_l^M]$ respectively. Under the VRS-technology specification, confinement operations exhibit high LMS proportions (say, 0.300 or above) for crop, animal, and machinery. Similarly, grazing operations show high LMS proportions for labor, animal, and crop acreage. The results under WDEA are generally similar, suggesting high LMS proportions for confinement's animal and grazer's labor and animal inputs. Some of the over-utilization may be caused by the medium to long term investments in production assets and contractual agreements or the upward bias from subsidized dairy or crop production. Turning to under-utilization of inputs, the VRS results find high LMD proportions for cow and crop acreage among confinement operations and no high LMD proportions among grazing operations. The WDEA results suggest high LMD proportions for labor and pasture acreage among confinement and a high LMD proportion for cow among grazers. Thus, the results indicate some under-utilization of dairy cows under both dairy systems. The producer may not expand his herd when operational capacity is nearly full, cows of desired characteristics are scarce in the market, or operation expansions are put on hold for idiosyncratic reasons (e.g., uncertainty for family labor supply). Crop acreage appears over-utilized by grazers and under-utilized by confinement perhaps due to the inefficiency in land markets.

Additionally, the sensitivity of LMS and LMD proportions are summarized as unit-free elas-

ticity measures with respect to market prices, or $|(dq/dp)/(q^M/p^M)|$ for LMS or LMD proportion q^M, market price p^M, and local slope dq/dp.[17] In contrast to the ideal factor market characterized with zero-LMS and zero-LMD with arbitrarily large elasticities, a poorly functioning factor market would exhibit a high LMS or LMD proportion with a very small elasticity. Panels B1 and B2 in table 3 report the calculated price elasticities of LMS and LMD across inputs and technology specifications. Among the inputs with high LMS proportions, highly-inelastic LMS (i.e. less than 0.300) are found in machinery (under VRS) for confinement and labor (under both VRS and WDEA) and crop acreage (under VRS) for grazers. These inputs appear systematically over-utilized since the extents of over-utilization depend little on their market rates. Capital-intensive confinement operations may face the difficulty in reversing their investments into machinery. Meanwhile, the apparent (and inexpensive) use of excess labor among grazers may rather increase profits if it is related to organic milk production for sufficiently high price premiums. Among the inputs with high LMD proportions, highly-inelastic LMD are identified for cows (under VRS) and crop acreage (under VRS and WDEA) among confinement and for cows (under WDEA) among grazers. It may be worthwhile to investigate potential market failures or input distortions that prevent these dairies from expanding their herds and/or systematically hinder transfers of crop acreage from grazers to confinement operators.

4 Conclusion

This article has developed a simple methodology that integrates the concepts of technical, scale, and allocative efficiencies into a nonparametric, technological-frontier estimation. The proposed Weighted DEA (WDEA) extends the standard VRS technological feasibility by estimating an optimal weighted average of the VRS, CRS, and profit frontiers. The proposed optimal weights minimize the sum of residual squares by regressing some initial adjustments for the VRS estimator on the conventional measures of scale and allocative inefficiencies. In the application to Maryland dairy data, the technical efficiency is on average 5.2% to 7.8% lower under WDEA, compared to the standard VRS estimates. Estimated producer-specific shadow values along the VRS and WDEA frontiers are generally similar with no obvious patterns of systematic differences. Considerations for alternative weight selection rules and rigorous statistical inferences are left for future research.

[17] The slope is estimated by a linear regression with observations restricted to those of non-zero shadow values falling within the ±5 percentile margins around the market price.

References

Allen, R., A. Athanassopoulos, R. Dyson, and E. Thanassoulis. 1997. "Weights restrictions and value judgements in Data Envelopment Analysis: Evolution, development and future directions." *Annals of Operations Research* 73:13–34.

Allen, R., and E. Thanassoulis. 2004. "Improving envelopment in data envelopment analysis." *European Journal of Operational Research* 154:363–379.

Banker, R.D., V.M. Gadh, and W.L. Gorr. 1993. "A Monte Carlo comparison of two production frontier estimation methods: Corrected ordinary least squares and data envelopment analysis." *European Journal of Operational Research* 67:332–343.

Brissimis, S.N., M.D. Delis, and E.G. Tsionas. 2010. "Technical and allocative efficiency in European banking." *European Journal of Operational Research* 204:153–163.

Chambers, R.G., Y. Chung, and R. Färe. 1998. "Profit, directional distance functions, and Nerlovian efficiency." *J. Optim. Theory Appl.* 98:351364.

Chambers, R.G., Y. Chung, and R. Fre. 1998. "Profit, directional distance functions, and Nerlovian efficiency." *J. Optim. Theory Appl.* 98:351364.

Chambers, R.G., and R. Färe. 2008. "A calculus for data envelopment analysis." *Journal of Productivity Analysis* 30:169–175.

Chung, Y., R. Fre, and S. Grosskopf. 1997. "Productivity and Undesirable Outputs: A Directional Distance Function Approach." *Journal of Environmental Management* 51:229–240.

Dyson, R., R. Allen, A. Camanho, V. Podinovski, C. Sarrico, and E. Shale. 2001. "Pitfalls and protocols in DEA." *European Journal of Operational Research* 132:245–259.

Dyson, R.G., and E. Thanassoulis. 1988. "Reducing Weight Flexibility in Data Envelopment Analysis." *The Journal of the Operational Research Society* 39:563–576.

Hanson, J., D. Johnson, E. Lichtenberg, and K. Minegishi. 2013. "Competitiveness of management-intensive grazing dairies in the mid-Atlantic region from 1995 to 2009." *Journal of Dairy Science* 96:1894–1904.

Khalili, M., A. Camanho, M. Portela, and M. Alirezaee. 2010. "The measurement of relative efficiency using data envelopment analysis with assurance regions that link inputs and outputs." *European Journal of Operational Research* 203:761–770.

Kneip, A., L. Simar, and P.W. Wilson. 2008. "Asymptotics and Consistent Bootstraps for Dea Estimators in Nonparametric Frontier Models." *Econometric Theory* 24:1663–1697.

—. 2011. "A Computationally Efficient, Consistent Bootstrap for Inference with Non-parametric DEA Estimators." *Computational Economics* 38:483–515.

Kumbhakar, S.C. 1989. "Estimation of Technical Efficiency Using Flexible Functional Form and Panel Data." *Journal of Business & Economic Statistics* 7:253–258.

—. 1997. "Modeling allocative inefficiency in a translog cost function and cost share equations: An exact relationship." *Journal of Econometrics* 76:351–356.

Kumbhakar, S.C., and E.G. Tsionas. 2011. "Stochastic error specification in primal and dual production systems." *Journal of Applied Econometrics* 26:270–297.

Kumbhakar, S.C., and H.J. Wang. 2006. "Estimation of technical and allocative inefficiency: A primal system approach." *Journal of Econometrics* 134:419–440.

Kuosmanen, T. 2005. "Weak Disposability in Nonparametric Production Analysis with Undesirable Outputs." *American Journal of Agricultural Economics* 87:1077 –1082.

Kuosmanen, T., L. Cherchye, and T. Sipilinen. 2006. "The law of one price in data envelopment analysis: Restricting weight flexibility across firms." *European Journal of Operational Research* 170:735–757.

Kuosmanen, T., and T. Post. 2001. "Measuring economic efficiency with incomplete price information: With an application to European commercial banks." *European Journal of Operational Research* 134:43–58.

Luenberger, D.G. 1994. "Dual Pareto Efficiency." *Journal of Economic Theory* 62:70–85.

Nerlove, M. 1965. *Estimation and identification of Cobb-Douglas production functions*. Rand McNally.

Podinovski, V. 2004a. "Suitability and redundancy of non-homogeneous weight restrictions for measuring the relative efficiency in DEA." *European Journal of Operational Research* 154:380–395.

Podinovski, V., and E. Thanassoulis. 2007. "Improving discrimination in data envelopment analysis: some practical suggestions." *Journal of Productivity Analysis* 28:117–126.

Podinovski, V.V. 2004b. "Production trade-offs and weight restrictions in data envelopment analysis." *Journal of the Operational Research Society* 55:1311–1322.

—. 2004c. "Production trade-offs and weight restrictions in data envelopment analysis." *Journal of the Operational Research Society* 55:1311–1322.

Podinovski, V.V., and T. Kuosmanen. 2011. "Modelling weak disposability in data envelopment analysis under relaxed convexity assumptions." *European Journal of Operational Research* 211:577–585.

Ruggiero, J. 1998. "Non-discretionary inputs in data envelopment analysis." *European Journal of Operational Research*, pp. 461–469.

Sarrico, C., and R. Dyson. 2004. "Restricting virtual weights in data envelopment analysis." *European Journal of Operational Research* 159:17–34.

Scheel, H. 2001. "Undesirable outputs in efficiency valuations." *European Journal of Operational Research* 132:400–410.

Schmidt, P., and C.K. Lovell. 1980. "Estimating stochastic production and cost frontiers when technical and allocative inefficiency are correlated." *Journal of Econometrics Journal of Econometrics* 13:83–100.

—. 1979. "Estimating technical and allocative inefficiency relative to stochastic production and cost frontiers." *Journal of Econometrics Journal of Econometrics* 9:343–366.

Seiford, L.M., and J. Zhu. 2002. "Modeling undesirable factors in efficiency evaluation." *European Journal of Operational Research* 142:16–20.

Simar, L., A. Vanhems, and P.W. Wilson. 2012. "Statistical inference for DEA estimators of directional distances." *European Journal of Operational Research* 220:853–864.

Simar, L., and P. Wilson. 2011. "Inference by the m out of n bootstrap in nonparametric frontier models." *Journal of Productivity Analysis* 36:33–53.

Thanassoulis, E., and R. Allen. 1998. "Simulating Weights Restrictions in Data Envelopment Analysis by Means of Unobserved DMUs." *Management Science* 44:586–594.

Thanassoulis, E., M.C. Portela, and R. Allen. 2004. "Incorporating Value Judgments in DEA." In W. W. Cooper, L. M. Seiford, and J. Zhu, eds. *Handbook on Data Envelopment Analysis*. Springer US, vol. 71 of *International Series in Operations Research & Management Science*, pp. 99–138.

Thompson, R.G., L.N. Langemeier, C.T. Lee, E. Lee, and R.M. Thrall. 1990. "The role of multiplier bounds in efficiency analysis with application to Kansas farming." *Journal of Econometrics* 46:93–108.

Tracy, D.L., and B. Chen. 2004. "A generalized model for weight restrictions in data envelopment analysis." *Journal of the Operational Research Society* 56:390–396.

Yotopoulos, P.A., and L.J. Lau. 1973. "A Test for Relative Economic Efficiency: Some Further Results." *The American Economic Review* 63:214–223.

5 Tables and Figures

Table 1: Summary Statistics of Variables By Dairy System

			Distribution				
	Mean	S.D.	Min.	25th	50th	75th	Max.
Confinement							
Milk (cwt)	24,150	17,577	3,761	13,680	19,890	28,550	110,700
Cow (animal)	122	76	22	70	108	145	468
Labor†	35,020	50,010	0	2,064	18,500	45,540	281,300
Crop†	47,400	40,954	0	19,860	35,870	58,910	231,000
Animal†	181,200	129,666	32,390	97,320	137,900	210,100	746,900
Machinery†	148,900	101,620	22,060	87,530	129,700	182,100	806,300
Crop Acre (acre)	289	155	60	175	210	350	704
Pasture Acre (acre)	50	39	0	20	40	80	141
Grazer							
Milk (cwt)	12,440	5,573	2,670	9,467	11,550	14,550	42,950
Cow (animal)	87	29	37	70	81	97	195
Labor†	6,229	10,383	0	0	1,109	8,608	75,320
Crop†	18,250	17,932	0	6,188	12,240	24,770	107,200
Animal†	94,290	48,034	7,882	59,780	86,470	127,300	255,800
Machinery†	84,460	45,587	26,720	54,540	73,230	96,900	327,000
Crop Acre (acre)	132	108	0	30	150	180	600
Pasture Acre (acre)	152	60	53	96	130	207	280

1. Unbalanced panel data set on 1995-2009 contains 17 grazers and 29 confinement dairies with 5 dairies switching from confinement to grazing during the period, totaling 475 operation-year observations.

2. † Categorical expenses comprise the following: machinery ≡ custom hire + depreciation + fuel + rent + maintenance + utility, labor ≡ labor + employment benefit + pension, crop ≡ seed + chemicals + fertilizer, animal ≡ feed + veterinary services.

Table 2: Summary of Efficiency Scores

			Distribution				
	Mean	S.D.	Min	25th	50th	75th	Max
Confinement							
TE (VRS)	0.896	0.095	0.589	0.821	0.905	0.996	1.000
SE (VRS)	0.965	0.058	0.677	0.962	0.989	0.999	1.000
AE (VRS)	0.787	0.095	0.525	0.729	0.788	0.853	1.000
SMB	0.819	0.120	0.562	0.730	0.800	0.925	1.000
WDEA	0.826	0.085	0.570	0.769	0.820	0.885	1.000
Grazers							
TE (VRS)	0.931	0.097	0.640	0.871	0.987	1.000	1.000
SE (VRS)	0.914	0.117	0.440	0.880	0.960	1.000	1.000
AE (VRS)	0.717	0.120	0.485	0.629	0.696	0.801	1.000
SMB	0.877	0.111	0.640	0.782	0.903	0.983	1.000
WDEA	0.883	0.090	0.629	0.840	0.913	0.948	1.000

1. TE (VRS), SMB, and WDEA are technical efficiency estimates. SE and AE are scale and allocative efficiency estimates respectively.

2. Weights (α, β) for WDEA, obtained in linear regressions, are (0.349, -0.209) for confinement and (0.138, -0.020) for grazers.

Table 3: Latent Marginal Supply and Demand (LMS and LMD) Proportions and Their Elasticities

Group	Technology	Cow	Labor	Crop	Animal	Machi.	Crop A.	Past. A.
A1. LMS Proportion								
Confinement	VRS	0.035	0.248	0.331	0.462	0.525	0.086	0.162
Confinement	WDEA	0.064	0.236	0.252	0.366	0.264	0.086	0.143
Grazers	VRS	0.075	0.472	0.130	0.323	0.242	0.311	0.248
Grazers	WDEA	0.081	0.466	0.205	0.360	0.261	0.280	0.137
A2. LMD Proportion								
Confinement	VRS	0.303	0.268	0.019	0.029	0.032	0.328	0.162
Confinement	WDEA	0.201	0.360	0.096	0.096	0.057	0.357	0.334
Grazers	VRS	0.180	0.043	0.093	0.037	0.043	0.062	0.180
Grazers	WDEA	0.354	0.106	0.149	0.068	0.099	0.106	0.255
B1. Elasticity of LMS								
Confinement	VRS	0.892	0.267	0.740	1.012	0.279	0.122	0.041
Confinement	WDEA	0.658	0.199	0.889	1.273	2.173	0.437	0.019
Grazers	VRS	0.396	0.071	0.449	0.674	0.771	0.075	0.063
Grazers	WDEA	0.317	0.089	0.648	1.060	1.180	0.104	0.186
B2. Elasticity of LMD								
Confinement	VRS	0.097	0.194	0.270	2.013	0.747	0.044	0.014
Confinement	WDEA	0.415	0.261	0.477	2.078	1.478	0.091	0.011
Grazers	VRS	0.154	0.005	0.098	1.080	0.952	0.047	0.020
Grazers	WDEA	0.083	0.300	0.196	0.850	1.152	0.082	0.027

1. Elasticity measures around the market prices are calculated through $|(dq/dp)/(q^M/p^M)|$ where p^M, q^M are the market price and the LMS or MUP proportion associated with WTP and WTA curves respectively, and dq/dp is the local slope estimate of those curves near the market valuation. The slops are estimated by linear regressions with observations restricted to those within the $\pm$ 5 percentile margins around the market price.

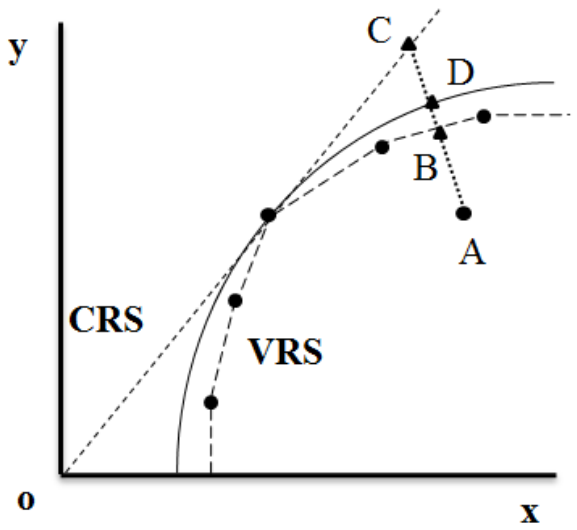

Figure 1: CRS, VRS, and Postulated Frontiers

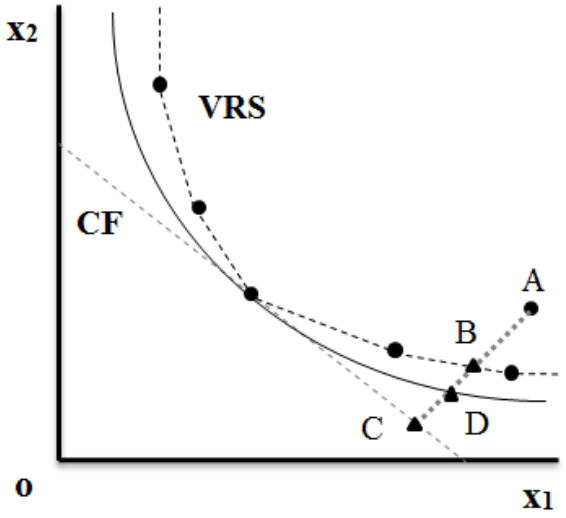

Figure 2: Cost, VRS, and Postulated Frontiers

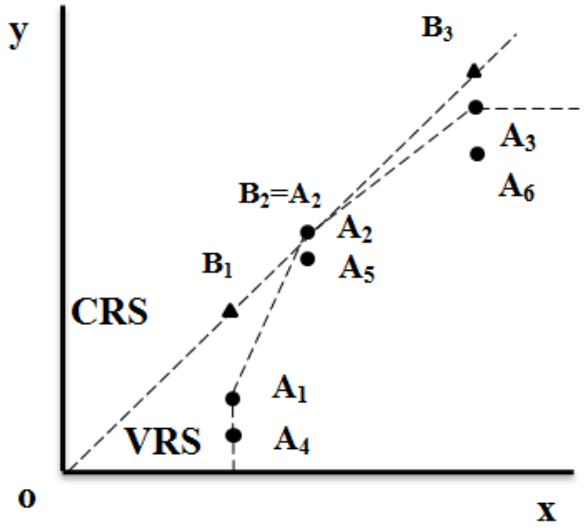

Figure 3: Example: SMB and WDEA (1/2)

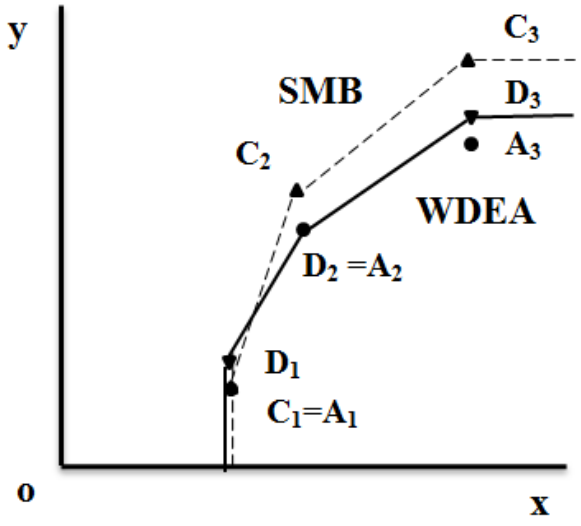

Figure 4: Example: SMB and WDEA (2/2)

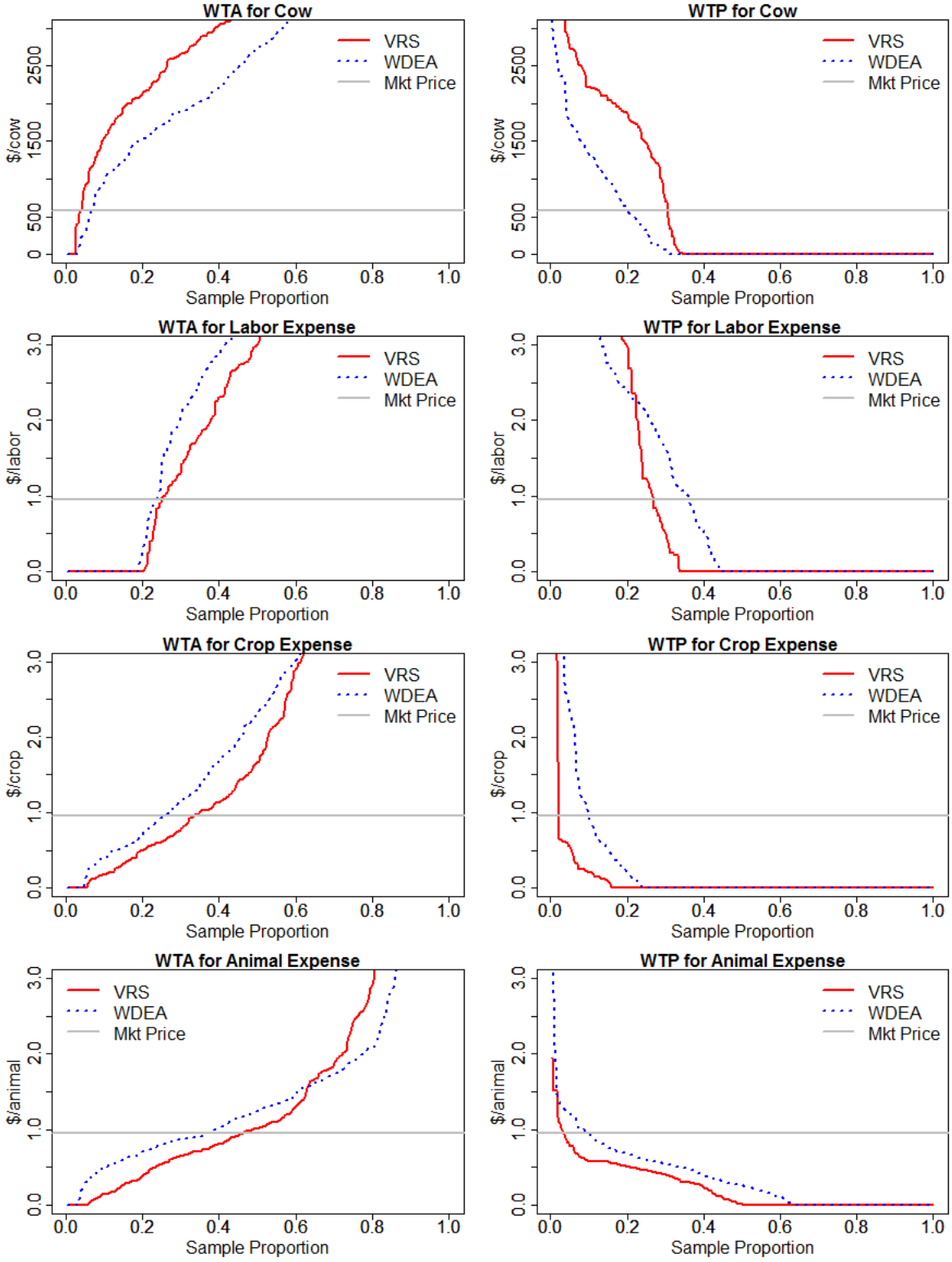

Figure 5: Shadow Values along VRS and WDEA Technological Frontiers (Confinement 1/2)

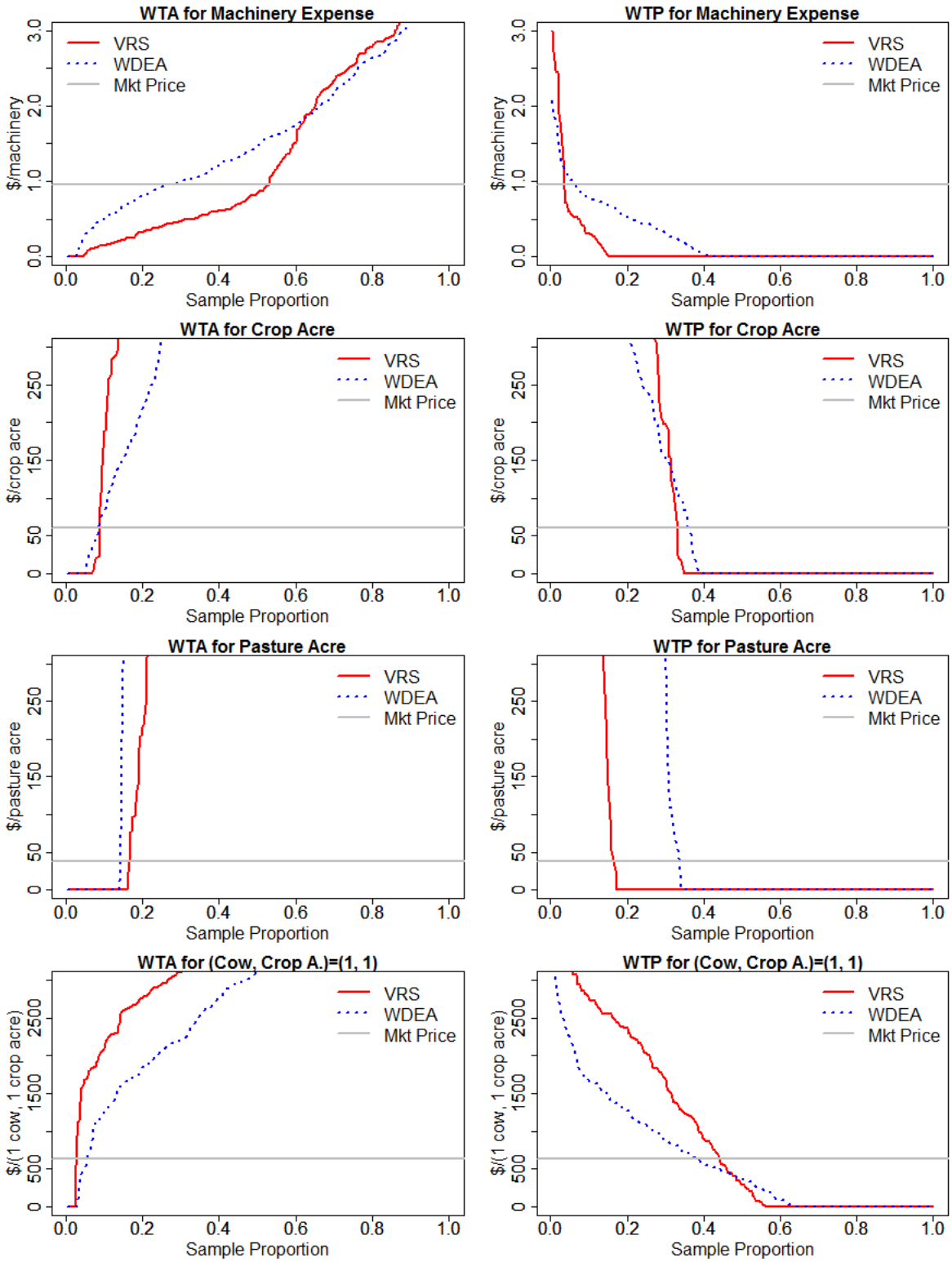

Figure 6: Shadow Values along VRS and WDEA Technological Frontiers (Confinement 2/2)

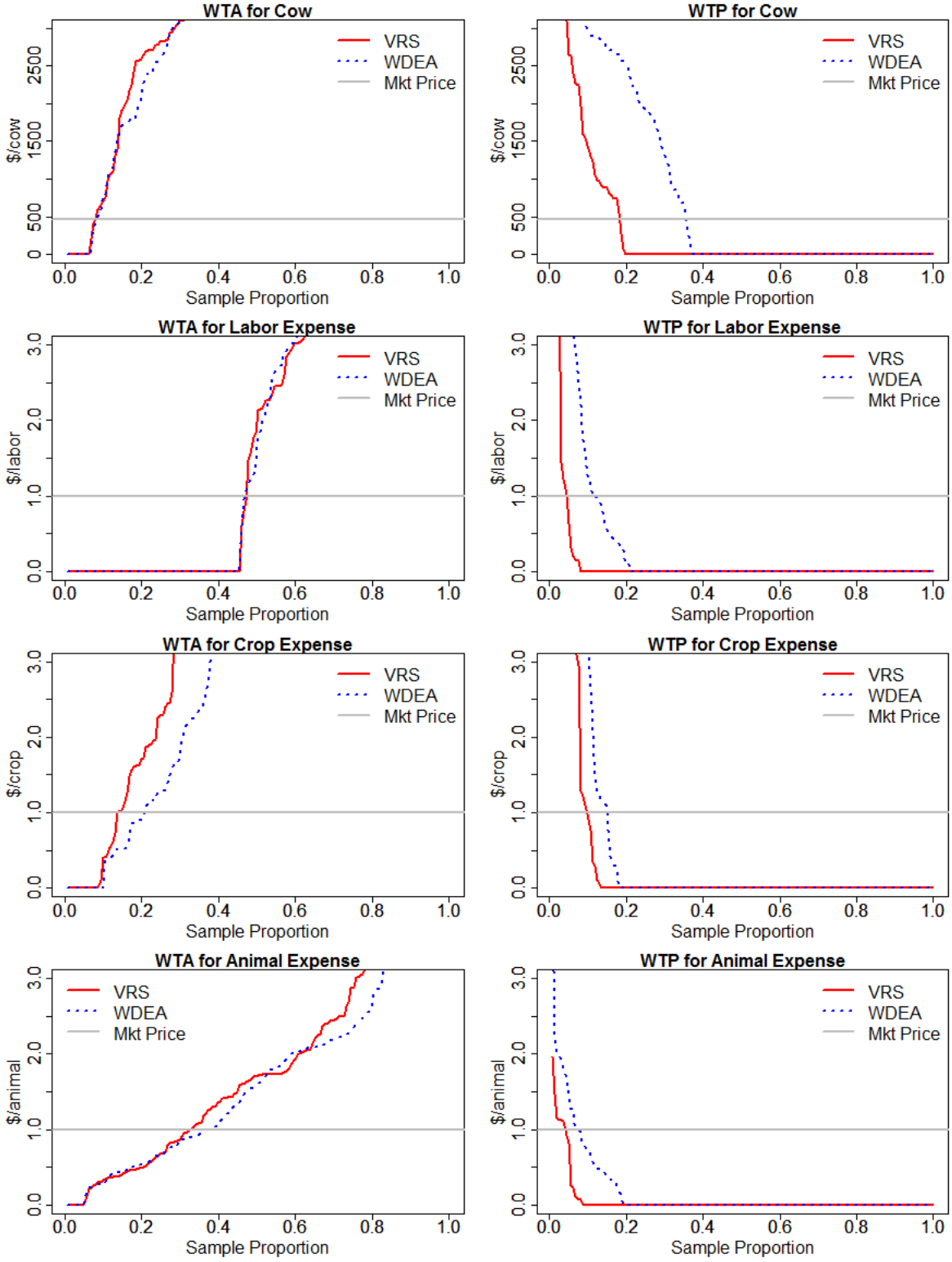

Figure 7: Shadow Values along VRS and WDEA Technological Frontiers (Grazers 1/2)

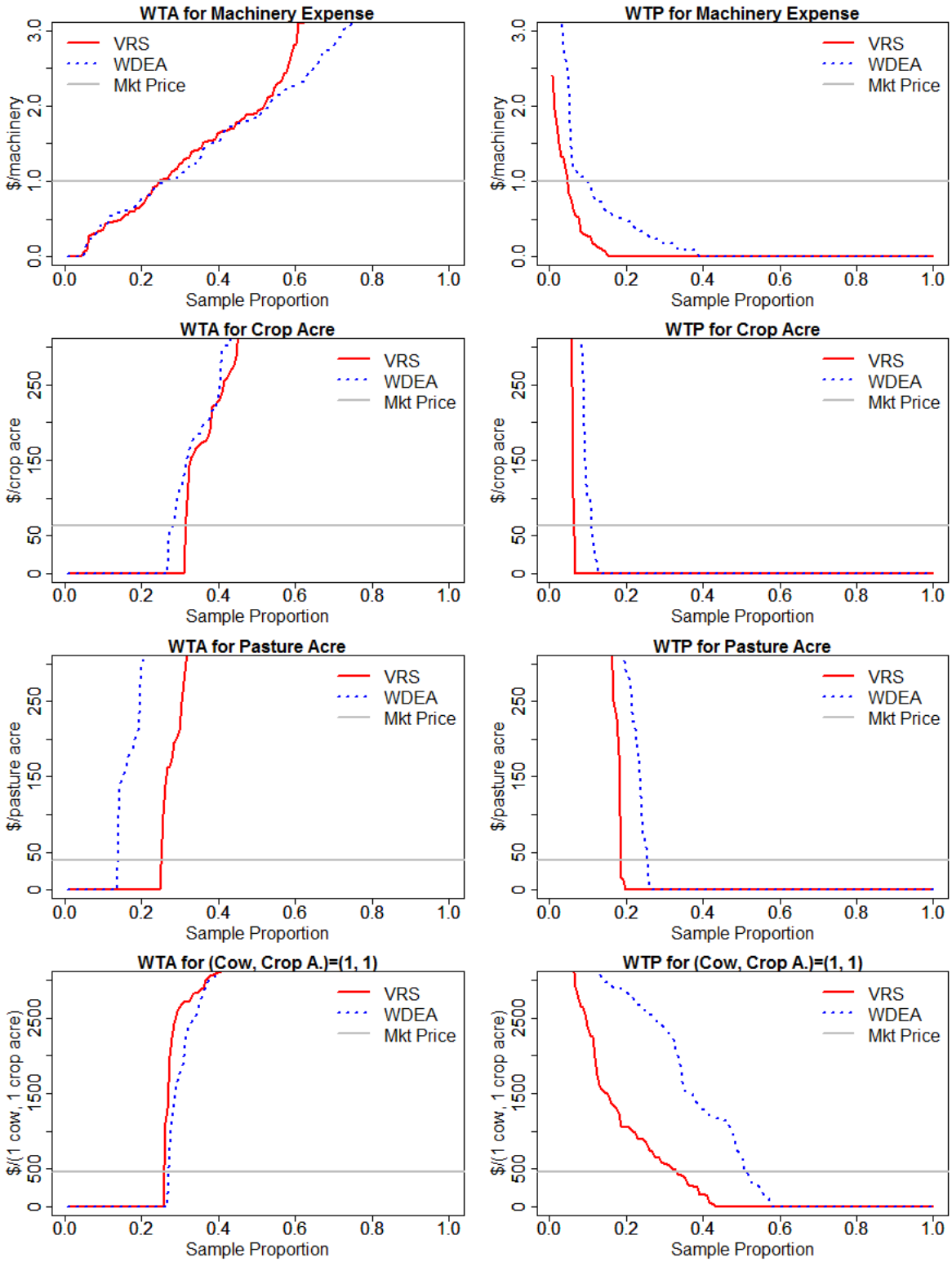

Figure 8: Shadow Values along VRS and WDEA Technological Frontiers (Grazers 2/2)

A Estimations of Market-level and Producer-level Shadow Prices

This supplementary section describes two procedures employed in the application section; one for estimating unknown market-level prices of inputs (which are used to calculate AI and constructing WDEA frontiers) and the other for estimating producer-level shadow values (which are used to characterize VRS and WDEA frontiers).

Market-level input prices are estimated as shared shadow-values in a DEA setting. The linear programming problem proposed by Kuosmanen, Cherchye, and Sipilinen (2006) combines (otherwise separately-estimated) DEA specifications of multiple producers into a single estimation problem for a common set of input prices that maximize the joint objective function subject to the (standard) DEA technological constraints;[18]

$$\max\{ \sum_{j\in\mathbb{I}} \gamma_j : \forall j \in \mathbb{I},\ \gamma_j \leq \boldsymbol{p}_j \boldsymbol{y}_j + f_j,\ \ \forall j,k \in \mathbb{I},\ \boldsymbol{p}_k \boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j + f_k \leq 0, \\ \boldsymbol{w}\left(\sum_{j\in\mathbb{I}} \boldsymbol{x}_j\right) \geq 1,\ \ \boldsymbol{p} \in \mathbb{R}_+^{MN},\ \boldsymbol{w} \in \mathbb{R}_+^{L},\ \boldsymbol{f} \in \mathbb{R}_+^{N}\} \tag{25}$$

where $\boldsymbol{p}_j$ is producer-specific shadow output values, f_j producer-specific scale parameter, and $\boldsymbol{w}$ the common input values across producers that are interpreted as market rates. To make the large scale linear programming problem manageable, this study estimates problem (25) as the average result of 100 subsample estimations, where each estimation uses 20 random observations in the sample.[19]

The current application additionally constraints the range of shadow values through incorporating market price information. Dairy production decision is modeled with milk output and the total of seven inputs including herd size, four categorical expenses (in labor, crop, animal, and machinery) divided by share-weighted price indices, and two types of land areas (for crop production and pasture). The common shadow-values of inputs are estimated by equation (25) with the following constraints;

C1. $w_{labor} = w_{crop} = w_{animal} = w_{machine}$

C2. $\forall i \in \mathbb{I},\ 0.90\ (p_{milk,i}/p^M_{milk}) \leq w_{labor} \leq 1.10\ (p_{milk,i}/p^M_{milk})$

C3. $w_{pasture_acre}/w^M_{pasture_acre} = w_{labor},\ \ w_{crop_acre}/w^M_{crop_acre} = w_{labor}$

C4. $0.5\, w_{cow(conf.)} \leq w_{cow(graz.)} \leq 1.5\, w_{cow(conf.)}$

C5. $0.90(\sum_i \pi_i/C_i)/N \leq (\sum_i - f_i)/N \leq 1.10\ (\sum_i \pi_i/C_i)/N.$

[18] The common input price under a free-disposable hall (FDH) is;
$\max\{\sum_j \gamma_j : \forall j,k, \gamma_j \leq \boldsymbol{p}_{jk}\boldsymbol{y}_j + f_{jk},\ \forall j,k, \boldsymbol{p}_{jk}\boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j + f_{jk} \leq 0, \boldsymbol{w}(\sum \boldsymbol{x}_j) \geq 1,\ \boldsymbol{p} \in \mathbb{R}_+^{MN^2},\ \boldsymbol{w} \in \mathbb{R}_+^{L},\ \boldsymbol{f} \in \mathbb{R}_+^{N^2}\}$.

[19] When drawing each random subsample, the ratio of two groups of dairy farms was fixed at that of the sample.

Item C1 sets an identical shadow value for categorical expenses in labor, crop, animal, and machinery since these variables are originally in dollar terms that should be valued equally. Item C2 normalizes the price level with nominal milk price p^M_{milk}=\$18.74/cwt, so that the shadow value for one hundredweight of milk is worth about 18.74 times of the dollar-valued expense within $\pm 10\%$ deviations. Similarly, item C3 normalizes the shadow rental rates for crop and pasture acreage by nominal rates $w^M_{pasture_acre}$ =\$63.35 and $w^M_{crop_acre}$ =\$39.02 per acre respectively, or the average rates in North Central Maryland during years 2008-2012.[20] In item C4, the rental rate for cow, estimated separately for two dairy systems of grazing and confinement, assumes the rate for grazers to fall within $\pm 50\%$ of the rate for confinemnt. Finally, item C5 confines the average shadow profit $\sum_i (\boldsymbol{p}_i \boldsymbol{y}_i - \boldsymbol{w}\boldsymbol{x}_i)/N$ to $\pm 10\%$ deviations from the sample-average returns to production costs. Once estimated, the shadow values can be converted back into dollar terms; for instance, the nominal shadow value for labor expenses is obtained as $w^N_{labor} = w_{labor} * (p^M_{milk}/\bar{p}_{milk,i})$ using the average shadow value $\bar{p}_{milk,i}$ of milk output.

Producer-specific shadow values are estimated as the marginal rate of transformations along a technological frontier. One difficulty is the non-uniqueness of the shadow values for inputs and outputs along a piecewise-linear DEA frontier, which admits infinitely many supporting hyperplanes at its kink points. Following Chambers and Färe (2008), a derivative-like concept for these hyperplanes are given by the supper differential of $D(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)$ for a change of inputs from $\boldsymbol{x}$ to $\boldsymbol{x}^c \in \mathbb{R}^L$;

$$\partial D(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) = \{\boldsymbol{\nu} \in \mathbb{R}^L_+ : D(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y) + \boldsymbol{\nu}(\boldsymbol{x}^c - \boldsymbol{x}) \geq D(\boldsymbol{x}^c, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y), \forall \boldsymbol{x}^c \in \mathbb{R}^L\}. \quad (26)$$

While any member of the super differential could be interpreted as shadow values of inputs and outputs, its directional derivative is uniquely defined as

$$D'_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{x}^c) = \lim_{\lambda \to 0+} \frac{D(\boldsymbol{x} + \lambda \boldsymbol{x}^c, \boldsymbol{y}, \boldsymbol{g}_x, \boldsymbol{g}_y) - D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{g}_x, \boldsymbol{g}_y)}{\lambda}, \quad (27)$$

which is positive linear homogeneous and concave in $\boldsymbol{x}^c$ and satisfies $D'_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{0}) = 0$.[21]

The most economically relevant shadow values are those for the willingness to pay (WTP) and the willingness to accept (WTA) for a marginal change in inputs. The authors show that under the directional change toward l-th unit vector $\boldsymbol{e}_l = [0..1..0]^T$ (i.e. $\boldsymbol{e}_l \boldsymbol{w} = w_l$), WTP for

[20] These rental prices for crop and pasture acres are taken from the mean rents, across counties in North Central Maryland and years 2008-2012, of the corresponding items in USDA-NASS rental rate estimates.

[21] It is a support function of the super differential; $D'_T(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y, \boldsymbol{x}^c) = \inf\{\boldsymbol{\nu}\boldsymbol{x}^c : \boldsymbol{\nu} \in \partial D(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{g}_x, \boldsymbol{g}_y)\}$.

input l at decision $(\boldsymbol{x}_0, \boldsymbol{y}_0)$ is calculated as;[22]

$$\begin{aligned}\min\{w_l : &\forall j \in \mathbb{I},\ \boldsymbol{p}\boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j + f \leq 0,\ \ \boldsymbol{w}\boldsymbol{g}_x \geq 1,\\ &\boldsymbol{p}\boldsymbol{y}_0 - \boldsymbol{w}\boldsymbol{x}_0 + f = 0,\ \ \boldsymbol{p} \in \mathbb{R}^M_+,\ \boldsymbol{w} \in \mathbb{R}^L_+,\ f \in \mathbb{R}\}\end{aligned} \tag{28}$$

where $\boldsymbol{g}_x$ is a bundle of inputs that normalizes shadow values (and $\boldsymbol{g}_x = \boldsymbol{x}_0$ for each observation 0 in this study). The estimated values are converted into dollar terms by, for example, $w^N_{labor,i} = w_{labor,i} * (p^M_{milk}/p_{milk,i})$ where $w_{labor,i}$ and $p_{milk,i}$ are the estimates from (28). Similarly, under directional change $-\boldsymbol{e}_l = [0..-1..0]^T$, willingness to accept (WTA) for input l is given by;[23]

$$\begin{aligned}\max\{w_l : &\forall j \in \mathbb{I},\ \boldsymbol{p}\boldsymbol{y}_j - \boldsymbol{w}\boldsymbol{x}_j + f \leq 0,\ \ \boldsymbol{w}\boldsymbol{g}_x \leq 1,\\ &\boldsymbol{p}\boldsymbol{y}_0 - \boldsymbol{w}\boldsymbol{x}_0 + f = 0,\ \ \boldsymbol{p} \in \mathbb{R}^M_+,\ \boldsymbol{w} \in \mathbb{R}^L_+,\ f \in \mathbb{R}\}.\end{aligned} \tag{29}$$

Table 4: Market Prices Used For Calculating Allocative Efficiency

	Cow	Labor	Crop	Animal	Machi.	Crop A.	Past. A.	Avg. Profit
Confinement	574.83	0.955	0.955	0.955	0.955	60.52	37.28	79,970
Grazer	463.51	1.000	1.000	1.000	1.000	63.36	39.03	55,335

1. Based on Kuosmanen, Cherchye, and Sipilinen (2006) with additional constraints.

B Relative Shadow Values Between VRS and WDEA: A Simple Case

Consider simple geometrical relationships of shadow values along VRS and WDEA frontiers, as depicted in figure 9. In two-input space x_1-x_2, several observations (shown as circles) are connected by a piecewise-linear VRS frontier. Observation B^{AE} is also allocatively efficient and supporting a cost frontier. For two observations B and B', the intersections between the radial contractions (toward the origin) and the cost frontier are denoted as points C and C' respectively. The figure shows that decisions B and B' use too much

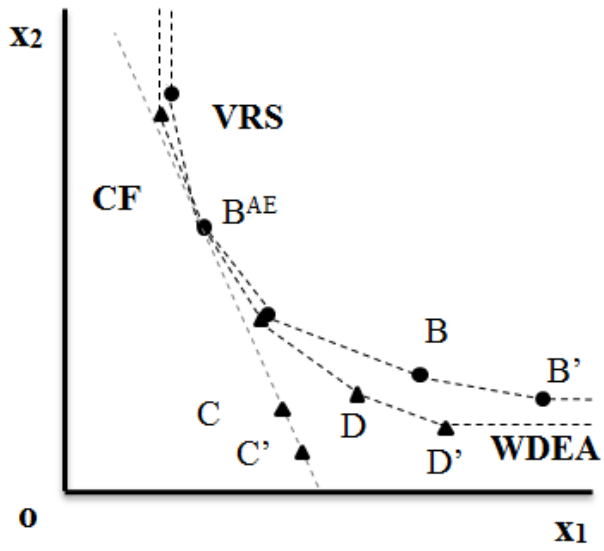

Figure 9: Shadow Values Along Frontiers

[22]Dual problem: $\max\{\theta : \sum_{j\in\mathbb{I}}\lambda_j\boldsymbol{y}_j \geq \mu\boldsymbol{y}_0,\ \sum_{j\in\mathbb{I}}\lambda_j\boldsymbol{x}_j \leq \mu\boldsymbol{x}_0 - \theta\boldsymbol{g}_0 + \boldsymbol{e}_l,\ \sum_{j\in\mathbb{I}}\lambda_j = \mu,\ \boldsymbol{\lambda}\mathbb{R}^N_+,\ \mu\mathbb{R}_+,\ \theta\mathbb{R}_+\}$.
[23]Dual problem: $\min\{\theta : \sum_{j\in\mathbb{I}}\lambda_j\boldsymbol{y}_j \geq \mu\boldsymbol{y}_0,\ \sum_{j\in\mathbb{I}}\lambda_j\boldsymbol{x}_j \leq \mu\boldsymbol{x}_0 + \theta\boldsymbol{g}_0 - \boldsymbol{e}_l,\ \sum_{j\in\mathbb{I}}\lambda_j = \mu,\ \boldsymbol{\lambda}\mathbb{R}^N_+,\ \mu\mathbb{R}_+,\ \theta\mathbb{R}_+\}$.

input x_1 and too little x_2, compared to decision B^{AE}. Thus, for example, at decision B the implied shadow value of input x_2 to input x_1 is lower than those of market rates, or the slope of BB' is smaller than the slope of CC'. Suppose that decision D between B and C and decision D' between B' and C' are the predicted technically-efficient decisions under WDEA with weights $\alpha > 0$ and $\beta = 0$. The slopes of segments BB', CC', and DD' correspond to the relative shadow values of input x_2 to input x_1 under the frontiers of VRS, cost, and WDEA respectively. The coordinates of decision $D = (x_1^D, x_2^D)$ are given by $x_l^D = x_l^B - \alpha(x_l^B - x_l^C) = (1-\alpha)x_l^B + \alpha x_l^C$ for input $l = 1, 2$. By nothing that $x_l^D - x_l^{D\prime} = (1-\alpha)(x_l^B - x_l^{B\prime}) + \alpha(x_l^C - x_l^{C\prime})$ for input $l = 1, 2$ on segment DD', it follows that the slope of BB' is smaller than the slope of DD';

$$\left|\frac{x_2^B - x_2^{B\prime}}{x_1^B - x_1^{B\prime}}\right| < \left|\frac{x_2^C - x_2^{C\prime}}{x_1^C - x_1^{C\prime}}\right| \quad \Rightarrow \quad \left|\frac{x_2^B - x_2^{B\prime}}{x_1^B - x_1^{B\prime}}\right| < \left|\frac{(1-\alpha)(x_2^B - x_2^{B\prime}) + \alpha(x_2^C - x_2^{C\prime})}{(1-\alpha)(x_1^B - x_1^{B\prime}) + \alpha(x_1^C - x_1^{C\prime})}\right|. \tag{30}$$

Similarly, if the slope of BB' were greater than the slope of CC', then the slope of BB' would be greater than the slope of DD'. This implies that the local marginal rate of substitution under WDEA (the slope of DD') is closer to the relative market rates (the slope of CC'), compared to that of VRS (the slope of BB'). Thus, WDEA's estimates for WTP or WTA of input x_1 at decisions D and D' are lower, and that of input x_2 higher than the VRS counterparts at decisions B and B'. While these results are fairly straightforward, such relationships become too complicated to derive simple characterizations in a higher dimensional input-output space.