ADVANCED AGRICULTURAL PRODUCTION ECONOMICS

by

DAVID L. DEBERTIN

Preface by the Author

I taught AEC 620, *Advanced Production Economics*, every year between 1974, when I arrived as an assistant professor at the University of Kentucky, to 2012, when I retired. The following publication is one of the early sets of class notes I used in the course, dating from 1978. These notes are what evolved into the hard-cover textbook "*Agricultural Production Economics*" Published by Macmillan in 1986. The 1986 text book evolved into the soft cover 2012 textbook "*Agricultural Production Economics Second edition*", available in bound paper volume on Amazon, at college bookstores, and as a free e-download at http://ageconsearch.umn.edu It is fun to look at these 1978 notes to see how my thoughts in production economics evolved over nearly 40 years.

If you found these notes on ageconsearch, do not miss looking at the 1978 publication *Computer Graphics - a Technique for the Analysis of Agricultural Production Functions* ageconsearch handle http://purl.umn.edu/159492 While you are at it, take a look at UK Staff Paper 303 titled *An Animated Instructional Module for Teaching Production Economics with the Aid of 3-D Graphics* ageconsearch handle http://purl.umn.edu/158806. While you are on staff paper 303 don't miss downloading and running the attached PowerPoint file.

If you are actually looking for the e-download of *Agricultural Production Economics Second Edition* go to http://purl.umn.edu/158319 The book of color illustrations can be found at http://purl.umn.edu/158320

Finally, make sure you download an e-copy of my new microeconomics text book *Applied Microeconomics: Consumption, Production and Markets* at http://purl.umn.edu/158321

David L. Debertin
Lexington, KY
October, 2013

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Department of Agricultural Economics
Special Report

ADVANCED AGRICULTURAL

PRODUCTION ECONOMICS

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July, 1978

A hierarchy of texts which deal with production economics.

1. Introductory

    McConnell, Campbell R., Economics, 5th Ed., McGraw Hill,
    New York, 1972.

    Samuelson, Paul A., Economics, 9th Ed., McGraw Hill, New
    York, 1973.

2. Junior Level

    I. Traditional Approach - (Graphical)

    Leftwich, Richard, The Price System and Resource Allocation,
    Holt, Rinehart and Winston, New York, 1966.

    Watson, D.S., Price Theory and Its Uses, Houghton Mifflen,
    Boston, 1968.

    Doll, John P., V. James Rhodes and Jerry G. West, Economics
    of Agricultural Production - Marketing and Policy, Richard
    D. Irwin, 1968.

    Ferguson, C.E., Microeconomic Theory, Richard D. Irwin,
    Homewood, Illinois, 1966. (Simple derivations using calculus
    in the footnotes.)

    II. Modern (Set Theory) Approach

    Walsh, V., Introduction to Contemporary Microeconomics,
    McGraw Hill, 1969. (Good introduction to modern theory. The
    preface to this book is especially interesting reading.)

3. First Year Graduate School

    I. Traditional Approach (Basic Calculus)

    Henderson, James M., and Richard E. Quandt, Microeconomic Theory,
    A Mathematical Approach, McGraw Hill, 1971.***(Intermediate steps
    in derivations which are not intuitively obvious are often left to
    the reader. At least initially, one spends a lot of time in the
    appendix.)

Carlson, Sune, A Study on the Pure Theory of Production, Reprints of Economic Classics, Sentry Press, New York, 1965. (Requires only basic calculus - complete and thorough notation often puts off the reader the 1st time through.)

Allen, R.G.D., Mathematical Economics, McMillan, London, 1956. (An earlier treatise using basic calculus.)

Baumol, William J., Economic Theory and Operations Analysis, Prentice Hall, Englewood Cliffs, New Jersey, 1965. (Ties together operations research techniques with economic theory quite nicely. A popular book with those studying for prelims. At a lower level of mathematical sophistication than H & Q.)

Heady, Earl O., Economics of Agricultural Production and Resource Use, Prentice Hall, Englewood Cliffs, N.J., 1952. (Heady's blue "tome" is a classic text dealing with applications of economics to agricultural production. No calculus is required. Extremely wordy but covers everything.)

4. Second and Third Year Graduate School

I. Traditional Approach (Advanced Calculus)

Samuelson, Paul Anthony, Foundations of Economic Analysis, Atheneum, New York, 1970, (paperback), Harvard University Press, 1947, (hardcover). (Classic treatise on applications of advanced calculus to economics - not to be confused with the introductory text. Written when Samuelson was a 22 year old graduate student at Harvard. Difficult to read unless one is extremely well grounded in advanced calculus and matrix algebra. Read the prefaces.)

Ferguson, C.E., The Neoclassical Theory of Production and Distribution, Cambridge Publishing, London, 1969.

Frisch, Ragnar, Theory of Production, Rand McNally, Chicago, 1965.

II. Modern Approach (Set Theory)

Quirk, James and Rubin Saposnik, Introduction to General Equilibrium Theory and Welfare Economics, McGraw Hill, New York, 1968. (This is at quite an advanced level, but is at least a start into the modern approach at the graduate level. Applications to agriculture have yet to be explored.)

CHAPTER I

AN INTRODUCTION

Production Economics

Production economics is concerned with the allocation of scarce resources amongst competing wants. It deals with a relationship

$$\text{Means} \rightarrow \text{Ends}$$

Means are physical resources, institutions and knowledge.

Ends are goals:

> profit maximization
>
> sales maximization
>
> cost minimization

Maximization and minimization pervade the science of economics.

<u>Mathematics</u> - convenient notation for expressing concepts dealing with fundamental principles of maximization or minimization.

<u>Graphics</u> - often easier to grasp concepts with graphs than with math. Graphics and mathematics are closely tied together.

Lots of concepts can be expressed with math that can't be expressed with graphs. We are limited with our graphs to 3 dimensions. With math, we can work with 4 or 5 or 6 or n dimensional Euclidean space. This becomes of major importance when we start looking at the rather common situation where an entrepreneur is using several different inputs to produce different outputs--the typical farmer's situation.

Not many journal articles are being published that have graphical rather than mathematical models. Mathematical models are concise--no place for fuzzy reasoning. To be learned in the profession requires an understanding of math. For many of you, this is your first exposure to mathematics in economics.

Production economics is, of course, concerned with the producer. Some have argued that everything else in the profession is a subset of production economics. This is probably true.

Consumption theory is strictly analogous to production theory.

Undergraduate course -- an indifference curve looks an awful lot
like an isoquant -- graphs are analogous.

The math is analogous too. This is convenient: once one has
grasped the concepts necessary to understand production theory,
consumption theory (price analysis) quite readily falls into
place (or, for that matter, vice versa).

Statics vs. Dynamics

This course:

- concerned with a static environment--time does not
  exist.

- Comparative statics - compare events at 2 points in
  time--no concern with the process of getting from one
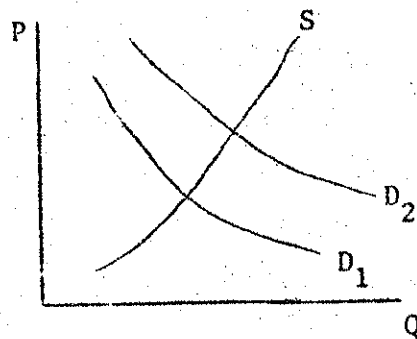 point to another. (Figure 1).

Figure 1. Illustration of comparative statics.

D_1 is observed in time period 1, D_2 in time period 2. We are not
concerned with the process of getting from D_1 to D_2--the process
is dynamics. (Figure 1).

Economists are often accused of relying too heavily on comparative
statics.

Perfect vs. Imperfect Knowledge

Perfect knowledge -- Entrepreneur knows everything including prices on
inputs and outputs. We assume perfect knowledge - often not really the
case. This assumption is not really relaxed until you get to AEC 621.

CHAPTER II
THE CONCEPT OF A PRODUCTION FUNCTION

1. Assume a (production) function

    (1) $y = f(x)$

    where y = an output   (dependent variable)

               x = an input   (independent variable)

    The function is nothing more than a <u>rule for assigning</u> to every value of x a <u>unique</u> (single) value of y.
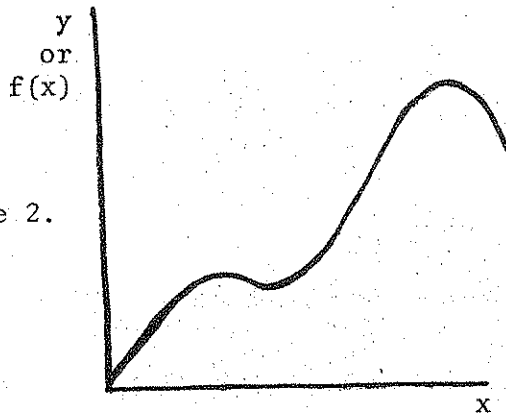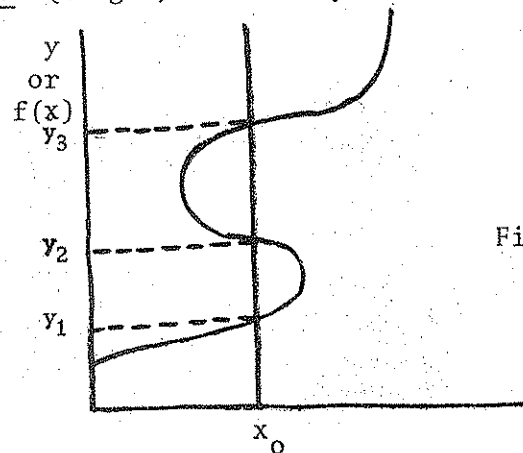

Figure 2.

Figure 3.

Figure 2 illustrates a function--for every value of x, a unique y value is assigned. Figure 3 is not a function. Note that when x assumes a value we call x_o, three possible values y_1, y_2 and y_3 are assigned to y.   <u>Figure 3 really depicts a correspondence rather than a function.</u>

Our function in Figure 2 is continuous--that is, there are no breaks in the function. Some functions are not continuous[1]. We often assume that production functions are continuous and twice differentiable. Most importantly, our function (1) describes a relationship - a technical input-output relationship. Our function (rule for assigning) states that for a given amount of input x, a <u>unique</u> amount of output y will be produced.

---

[1]For example: Suppose the <u>rule for assigning</u> is

        $y = 3$ if $x < 4$
        $y = 5$ if $x > 4$
    A function is continuous if

        $f(x_o)$ is defined $\lim_{x \to x_o} f(x)$ exists, and $\lim_{x \to x_o} f(x) = f(x_o)$

Note also the implicit assumption of technical efficiency. There is only one possible amount of output y that can be produced from a given amount of x. The modern therorests would refer to this as being "on the frontier."

Suppose our production function describes corn response to nitrogen fertilizer (x is fertilizer, y is bushels of corn harvested).

Really, lots of other inputs are used to produce corn. It is often convenient to hold other inputs constant. We might write this as:

$$y = f(x_1 \mid x_2, x_3, \ldots, x_n)$$

Where the slash denotes the word "given", and $(x_2, x_3, \ldots, x_n)$ represents a <u>vector</u> of inputs that are held constant at some predetermined <u>value</u>. (i.e., x_2' units of land, x_3'' units of labor)

Introduction to Maximization

Suppose the technical relationship describing corn response to nitrogen fertilizer looks like the relationship depicted in Figure 4.
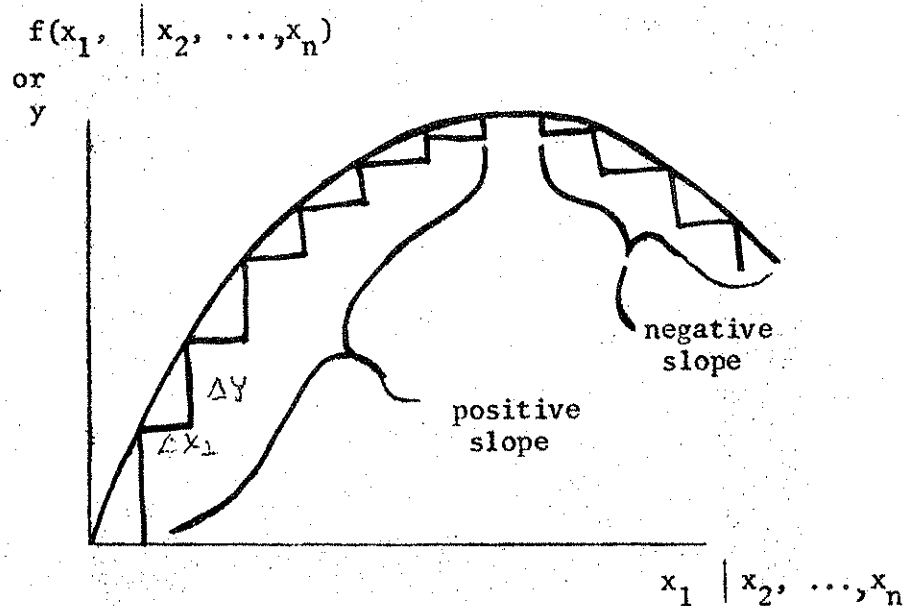

Figure 4. Corn response to nitrogen fertilizer

Suppose also we wish to maximize the amount of corn to be produced, forgetting for the moment that fertilizer costs money. It is obvious from Figure 4 that the maximum amount of corn can be produced where the slope of the function equals zero - that is:

where: $$\frac{\Delta y}{\Delta x} = 0$$

Δ (delta) is a mathematical <u>operator</u> denoting change

Since our function curves continuously (<u>has continuously curving tangents</u>) the slope of the function is never equal to zero over a finite change in x. No matter how small a change we assume in x, our function will always be turning slightly (Figure 5).
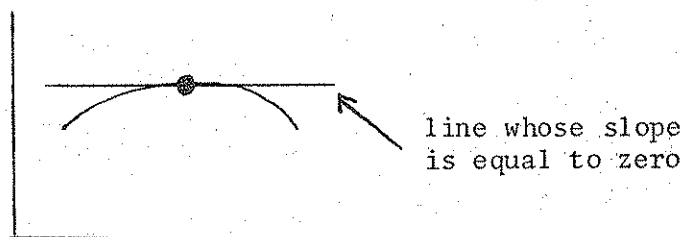

Figure 5. Magnification of the Top of our Production Function

If one assumes a change in x infinitely small, there will be a point where the slope of the production function is exactly equal to zero.

We use the symbol $\frac{dy}{dx}$ instead of $\frac{\Delta y}{\Delta x}$ to represent the slope of our production function when an infinitely small change in x is assumed.

$\frac{dy}{dx}$ is, of course, the <u>first derivative</u> of the function $y = f(x)$.

The derivative $\frac{dy}{dx}$ is in fact the <u>limit</u> of $\frac{\Delta y}{\Delta x}$ as Δx approaches zero. We write this as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

It is a necessary condition for maximization (of in this case, corn production) that the first derivative of the production

function be equal to zero.[2] In general, a necessary condition for the maximization of any function is that the first derivative is equal to zero.

From undergraduate economics courses, you probably remember that the slope of the production function $\frac{\Delta y}{\Delta x}$ is the marginal product of x, the addition to output associated with each additional unit of input. When marginal product is zero, total product is maximized.

It should now come as no shock that the first derivative of the production function, or total product function is the marginal product function.

$$y = TP \text{ or } TPP$$

$$\frac{\Delta y}{\Delta x} = MPP \text{ for a finite change in } x$$

$$\frac{dy}{dx} = MPP \text{ or the marginal productivity}$$
$$\text{function when } \Delta x \to 0$$

also: $f'(x) = MPP$ of the TPP function $f(x)$

Suppose: $y = f(x)$

the specific form of $f(x)$ is

$$-x^2 + 7x$$

Hence

$$TPP = -x^2 + 7x$$

$$MPP = -2x + 7$$

---

[2]This is a necessary but not sufficient condition for output maximization. We must also make certain that we are not at a point of output minimization rather than maximization.

The Concept Of An Integral
As Applied To Production Economics

Suppose we wish to estimate the area under the marginal physical

product function (curve)

$$MPP = -2x + 7$$

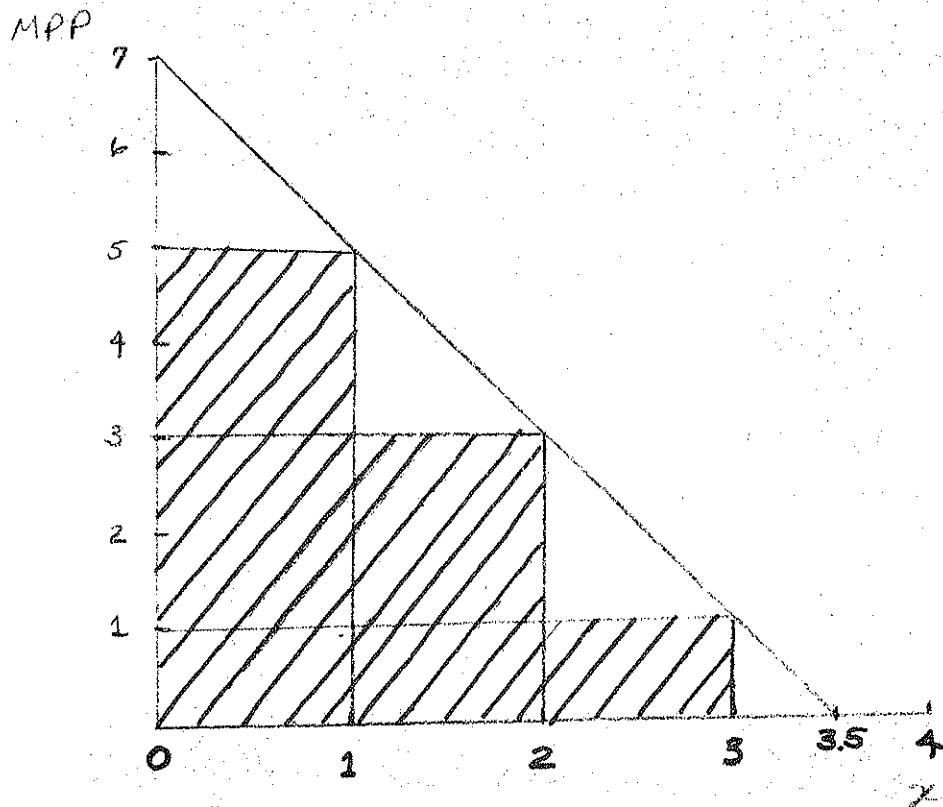We could approximate this area as suggested by the shaded area in

Figure 6.

Figure 6.

An _approximation_ of the area under the MPP function is:

$$(1) \quad 5(1-0) + 3(2-1) + 1(3-2)$$

which is an equation representing the shaded area in Figure 6.

Alternately this can be written as:

$$(2) \quad 5 + 3 + 1 = 8$$

or as:

$$(3) \quad \sum_{i=1}^{3} (-2x_i + 7)(x_i - x_{i-1})$$

and

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

Show that (3) is equivalent to (2) and (1).

Note also:

$$(4) \quad \sum_{i=1}^{3} (-2x_i + 7)(x_i - x_{i-1})$$

$$= \sum_{i=1}^{3} (-2x_i + 7) \Delta x_i$$

An integral is equivalent to the above sum when Δx_i is zero.

An integral is to a sum as a derivative is to a finite change. In other words:

$$(5) \quad \sum_{i=1}^{3} (-2x_i + 7) \Delta x_i \text{ as } \Delta x_i \to 0$$

is

$$(6) \quad \int (-2x_i + 7) \, dx_i$$

The integral (6) is exactly equal to the area under the MPP curve (TPP). The sum (4) is an approximation of the integral for finite values of Δx_i.

Evaluating The Area Under The Integral

Suppose we want an estimate of TPP as x goes from 2 to 4. A fundamental theorem for evaluating definite integrals applies to give us the formula:

$$\int_{2}^{4} MPPdx = TPP \text{ at } x = 4$$
$$-TPP \text{ at } x = 2$$

What theorem is being applied?

Since:

$$\int_{2}^{4} f'(x)dx = f(4) - f(2)$$

TPP at x = 4   -16 + 28 = 12

TPP at x = 2   - 4 + 14 = 10

Fundamental Truth

Assuming that TPP starts at the origin, the area under the MPP curve is TPP.

Chapter II

Exercise 1

1. Given the production function

    Corn = f(Nitrogen)

    what is the domain of the function?
    what is the range of the function?

2. What is the difference between a function and a correspondence?

3. How does $\frac{\Delta y}{\Delta x}$ differ from $\frac{dy}{dx}$?

4. State the meaning of $\frac{dy}{dx}$ for a production function relating

    corn response to nitrogen fertilizer.

5. Given the production function

    $$y = f(x)$$

    What is MPP ?

    APP ?

    TPP ?

6. If MPP is defined as

$$\frac{df}{dx},$$

how would one go about finding TPP when x assumes a fixed value x*? What fundamental integration theorem is used to solve this problem?

7. How does $\int x^2 dx$ differ from $\Sigma\ x^2 \Delta x$?

8. What are the assumptions of pure competition?

Increasing, Constant and
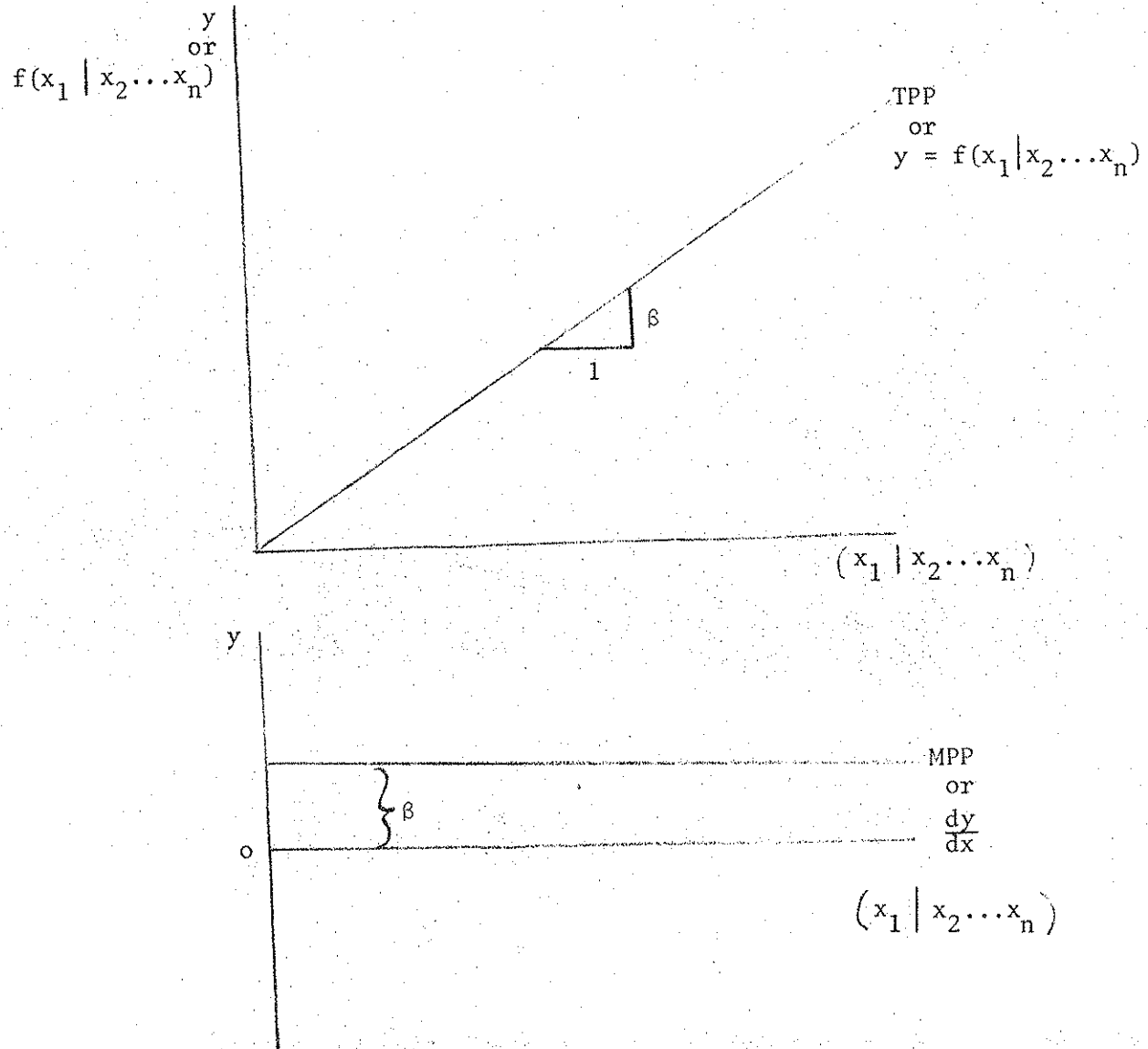
Diminishing (Marginal) Returns

1. We have examined a case in which there is <u>diminishing marginal returns</u> to the variable input - (nitrogen fertilizer).

The law of diminishing returns - "As we add units of nitrogen fertilizer to corn, after a point each additional unit of fertilizer produces less and less additional output." This law is, of course, fundamental to all of production economics.
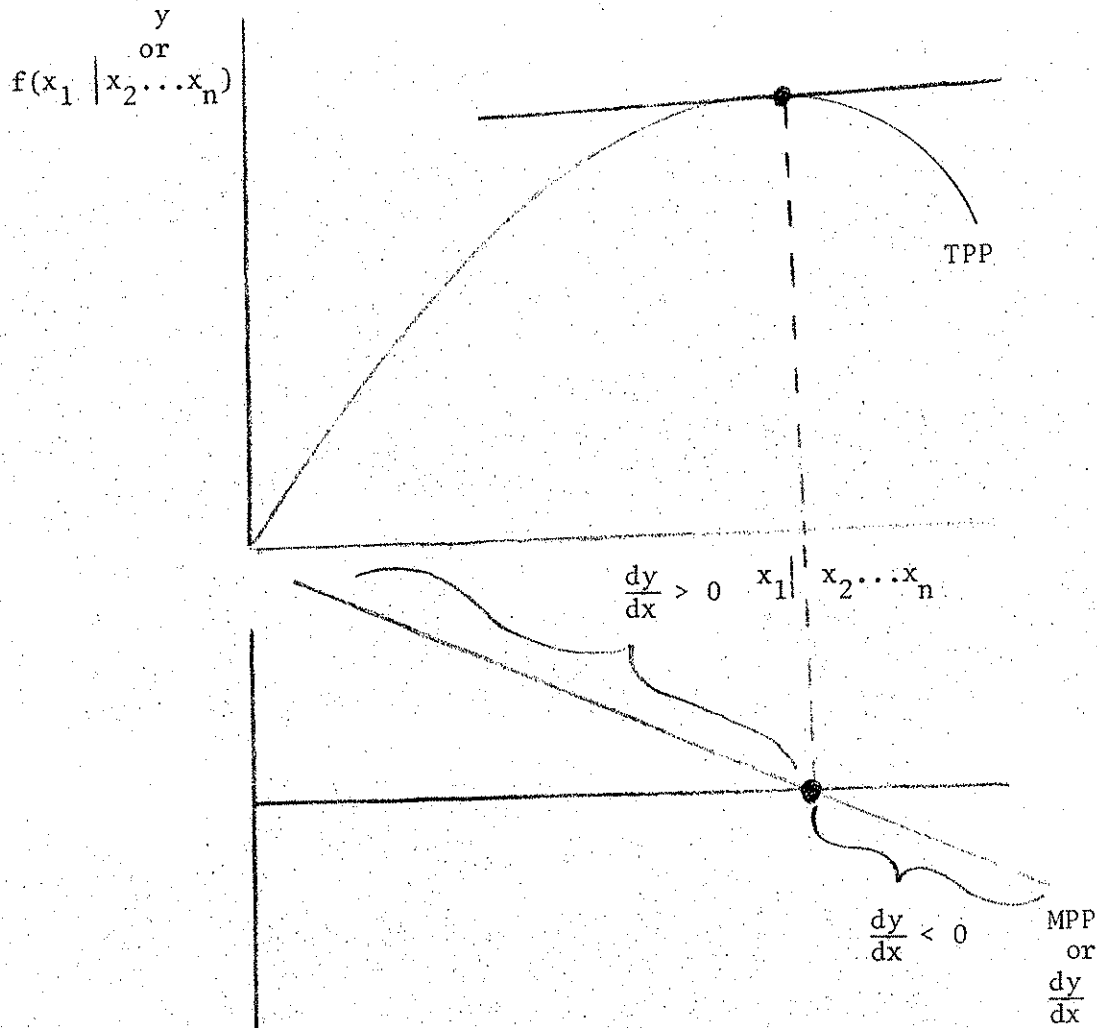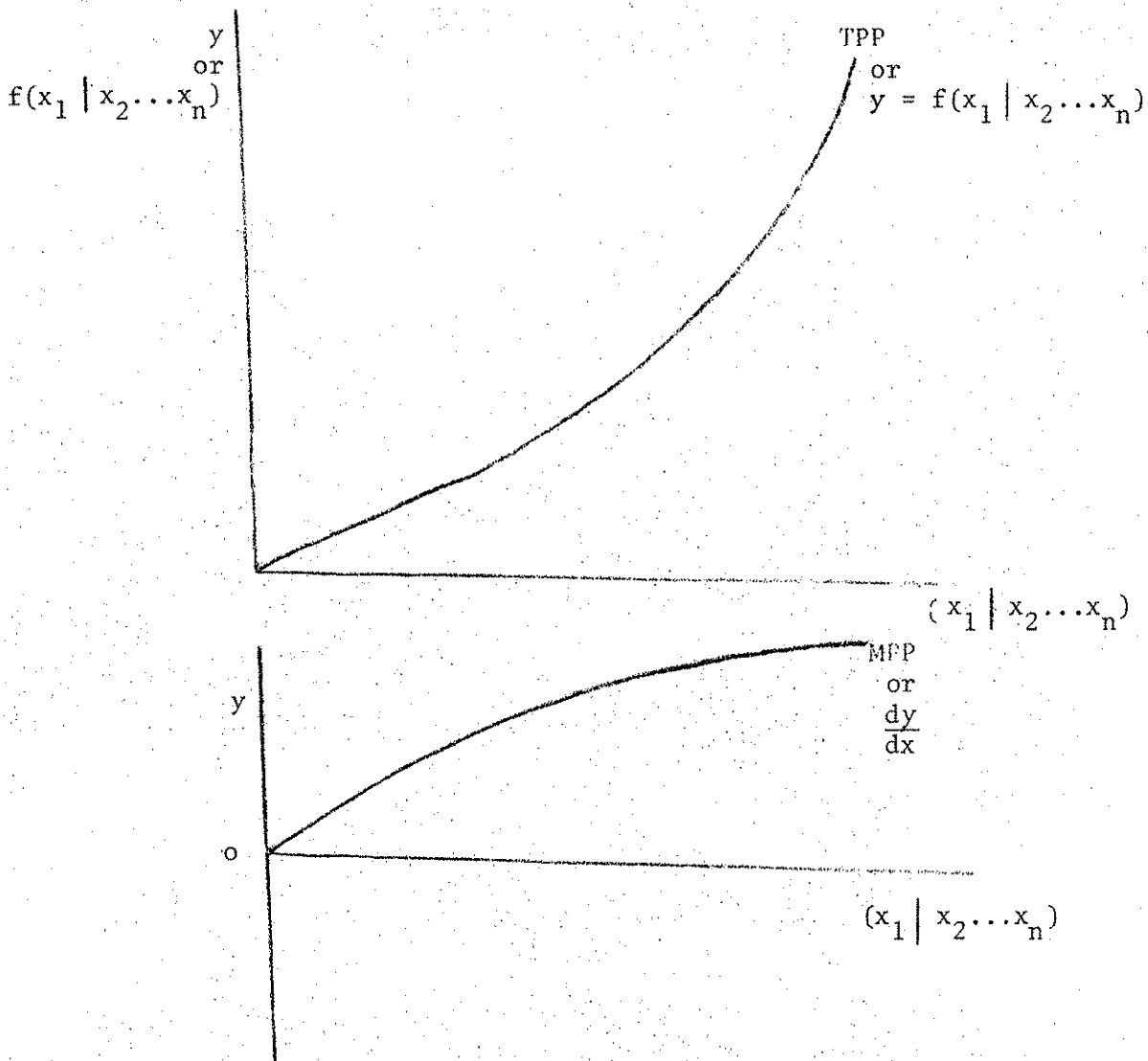

Note that the function $\frac{dy}{dx}$ has a negative slope. Each <u>additional</u> unit of x is producing less and less <u>additional</u> y.

2. Constant (marginal) returns to the variable input

$$\frac{dy}{dx} = \beta \text{ where } \beta \text{ is the slope of the TPP curve}$$

3. Increasing (marginal) returns to the variable input

$$y \text{ or } f(x_1 \mid x_2 \ldots x_n)$$

TPP or $y = f(x_1 \mid x_2 \ldots x_n)$

$(x_1 \mid x_2 \ldots x_n)$

y

MPP or $\dfrac{dy}{dx}$

o

$(x_1 \mid x_2 \ldots x_n)$

If the total product curve comes out of the origin the area under the MPP function is exactly equal to total product.

This follows because:

$$\int_{x=o}^{x = x_o} f'(x)dx = f(x)_{x=x_o} - f(x)_{x=o}$$

What if the total product curve does not come out of the origin?

Average Physical Product

APP or AP

$$APP = \frac{y}{x}$$

Amount of output produced per unit of input

Relationship Between APP and MPP

(1) $\quad y = f(x)$

(2) $\quad y = \frac{y}{x} \cdot x$

(3) $\quad y = \bar{y} \cdot x$

(4) $\quad \frac{dy}{dx} = \bar{y} + x \frac{d\bar{y}}{dx}$

Hence:

$$MPP = APP + x \frac{dAPP}{dx}$$

then, if

$$\frac{dAPP}{dx} > 0 , \qquad MPP > APP$$

$$\frac{dAPP}{dx} = 0 , \qquad MPP = APP$$

$$\frac{dAPP}{dx} < 0 , \qquad MPP < APP$$

Marginal Product vs. Marginal Productivity

The marginal productivity function is the derivative of the total produce function

$$MPP = \frac{dy}{dx} = \frac{df(x)}{dx}$$

and is a function representing the <u>rate of change</u> in total product.
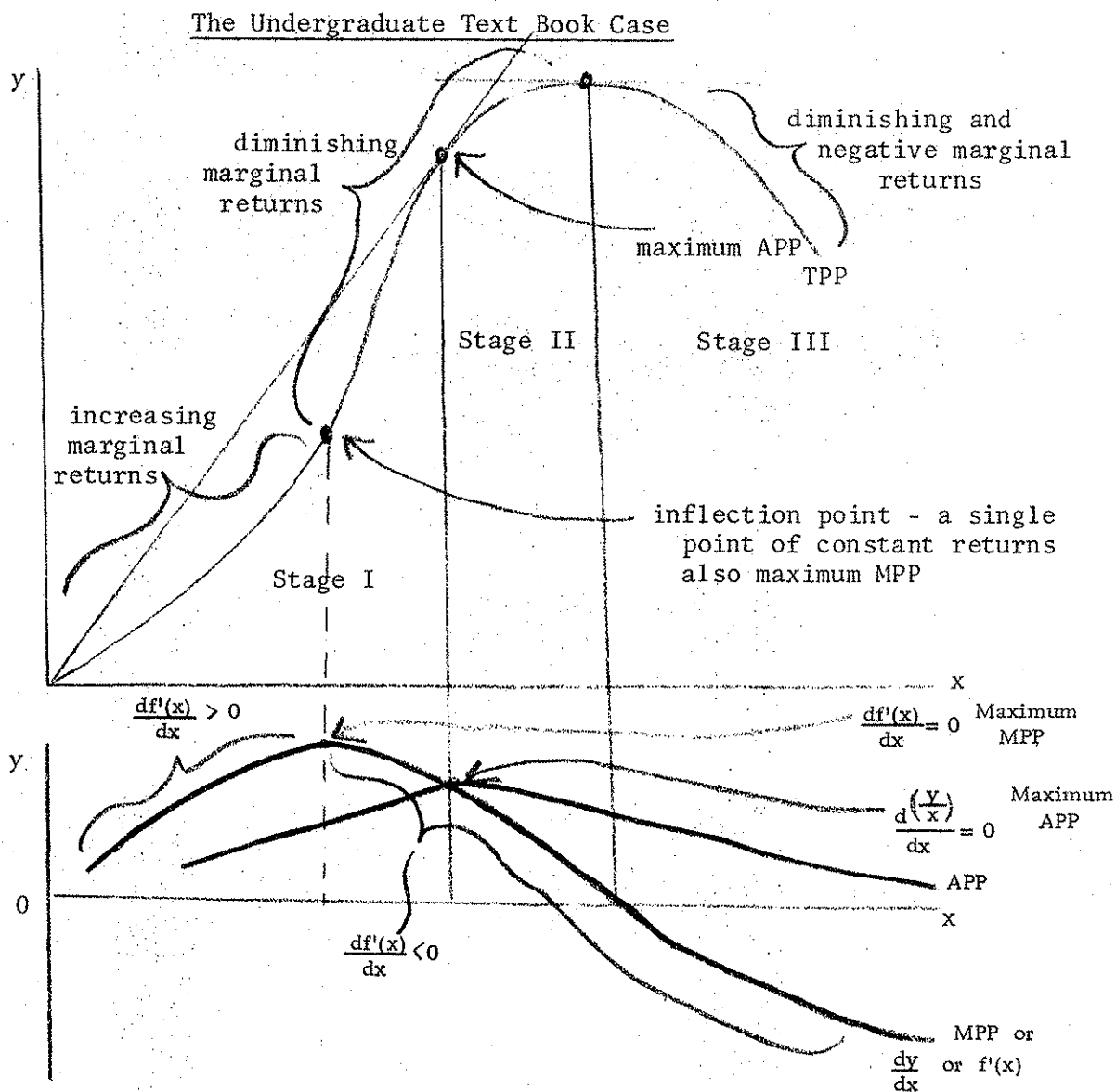Taking the differential of

$$y = f(x)$$

yields

$$dy = \frac{df(x)}{dx} \, dx$$

which is the marginal product dy of the increment dx, or in finite
notation

$$\Delta y = \frac{df(x)}{dx} \, \Delta x$$

where

$\frac{df(x)}{dx}$ is a <u>slope coefficient</u> (not necessarily constant,
but not at all unlike a regression coefficient)
representing the marginal productivity function.

The Undergraduate Text Book Case

Production Elasticities

Define:

$$\epsilon = \text{an elasticity of production}$$

$$= \frac{\% \; \Delta \; \text{in} \; y}{\% \; \Delta \; \text{in} \; x} =$$

$$\frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} \approx$$

$$\frac{dy}{dx} \cdot \frac{x}{y} \quad \text{or} \quad \frac{\frac{dy}{y}}{\frac{dx}{x}}$$

Exercise

Show that

$$\epsilon > 1 \quad \text{in stage I}$$
$$0 < \epsilon < 1 \quad \text{in stage II}$$
$$\epsilon < 0 \quad \text{in stage III}$$
$$\epsilon = 0 \quad \text{at beginning of Stage III}$$

Hint:

note that $\quad \epsilon = \frac{MPP}{APP}$

ϵ can also be interpreted as the "scale parameter."

$\epsilon > 1$ increasing returns to scale

$\epsilon = 1$ constant returns to scale

$\epsilon < 1$ diminishing returns to scale

Keep in mind that, in the textbook case ϵ is continually changing along the production function. This is because the ratio of MPP to APP is continually changing.

For a multiplicative power production function such as:

$$y = A x^{\alpha} \qquad \varepsilon = \alpha$$

$$\text{or} \qquad y = A x_1^{\alpha} x_2^{\beta} \qquad \varepsilon = \alpha + \beta$$

$$\text{or} \qquad y = A x_1^{.3} x_2^{.5} \qquad \varepsilon = .8$$

$$\text{or} \qquad y = A x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

$$\varepsilon = \sum_{i=1}^{n} \alpha_i$$

Merely sum the superscripts to calculate ε.

Some characteristics of textbook (graphical) production functions. Can you think of others?

(1) MPP increasing if marginal product is increasing

(2) MPP is at the maximum at the inflection point of the TPP curve

(3) MPP crosses APP at APP maximum

(4) Stage II starts where APP is maximum

(5) MPP = 0 when TPP is maximum

(6) MPP < 0 when TPP is declining

(7) Stage III starts where TPP is maximum and MPP is zero

These same characteristics stated with mathematics. What others exist?

(1) $\dfrac{d^2y}{dx^2} > 0$ implies increasing marginal product.

(2) $\dfrac{d^2y}{dx^2} = 0$ at inflection point of TPP curve.

(3) when $\dfrac{d\left(\dfrac{y}{x}\right)}{dx} = 0,$ $\dfrac{dy}{dx} = \dfrac{y}{x}$

(4) $\dfrac{d\left(\dfrac{y}{x}\right)}{dx} = 0$ where Stage II starts

(5) $\dfrac{dy}{dx} = 0$ when $f(x)$ is maximum

(6) $\dfrac{dy}{dx} < 0$ when $f(x)$ is declining

(7) the point where $\dfrac{dy}{dx} = 0$ marks the beginning of Stage III

Exercise 2

1. Suppose a production function

$$y = f(x)$$

has the derivatives

$$\frac{dy}{dx} = \frac{df}{dx} < 0$$

$$\frac{d^2y}{dx^2} = \frac{d^2f}{dx^2} < 0$$

$$\frac{d^3y}{dx^3} = \frac{d^3f}{dx^3} < 0$$

Draw the TPP and MPP curve corresponding to these conditions.

2. Suppose a production function

$$y = 2x$$

Draw the TPP and MPP corresponding to this function.

3. Suppose a production function

$$y = x^{.7}$$

Draw the TPP, MPP and APP corresponding to this function.

4. Suppose a production function

$$y = x^{1.5}$$

Draw the TPP, MPP and APP corresponding to this function.

5. Suppose a production function

$$y = x^{2.5}$$

Draw the TPP, MPP and APP corresponding to this function.

6. Suppose a production function

$$y = x^{.5}$$

How much will y increase if x is initially at 16 units and is then increased by 2 units?

CHAPTER III

PROFIT MAXIMIZATION:
THE SINGLE VARIABLE CASE

Suppose that:
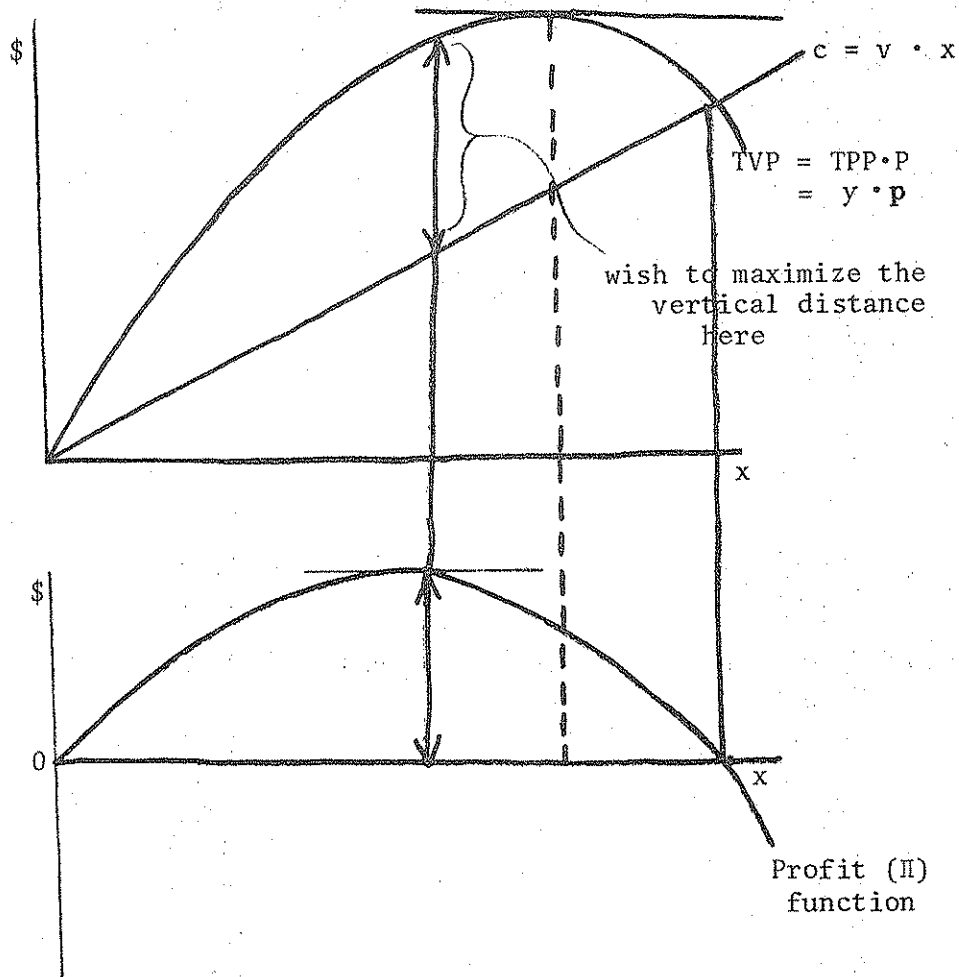
$$\Pi = p \cdot y - v \cdot x$$

where:

    y = output

    p = price of output

    x = input

    v = price of input

    Π = profit

$c = v \cdot x$

$TVP = TPP \cdot P$
$\quad = y \cdot p$

wish to maximize the
vertical distance
here

Profit (Π)
function

We wish to find a point on profit function where slope equals zero.

$$\Pi = py - vx$$

$$\frac{d\Pi}{dx} = p\,\frac{dy}{dx} - v = 0$$

$$p\,\frac{dy}{dx} = v$$

$$p \cdot MPP = v$$

$$MVP = MRC = \text{price of input}$$

The first order condition for a <u>maximum</u> is a <u>necessary</u> but <u>not sufficient</u> condition.

How do we know we have not found a point where profits are minimized rather than maximized?

The <u>second order condition</u> is the <u>second derivative test</u> for a maximum.

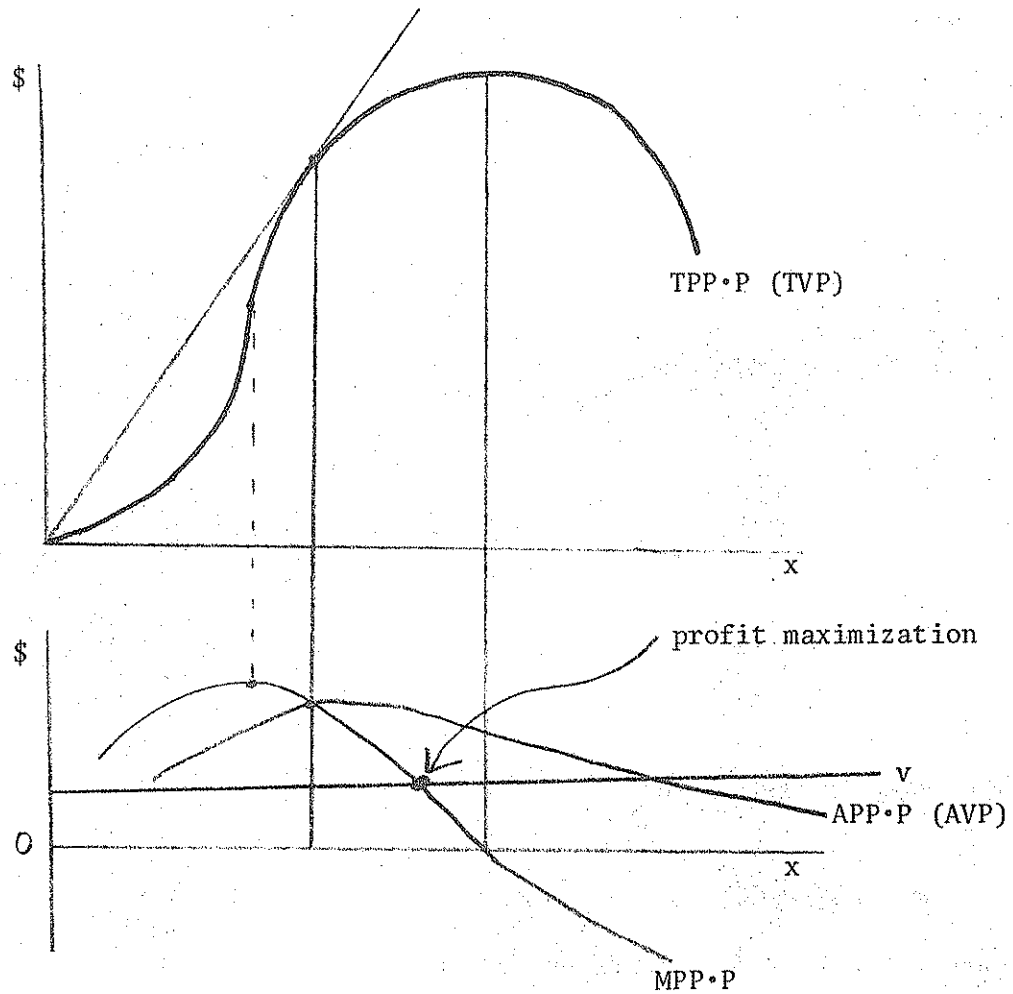$$\frac{d\left(\frac{d\Pi}{dx}\right)}{dx} < 0$$

for a minimum

$$\frac{d\left(\frac{d\Pi}{dx}\right)}{dx} > 0$$

$$\frac{d\left(\frac{d\Pi}{dx}\right)}{dx} = \frac{d^2\Pi}{dx^2}$$

First order condition is a <u>necessary condition</u> - without the condition the event will never occur.

Second order condition is a <u>sufficient condition</u> - with the condition insures that the event will occur.
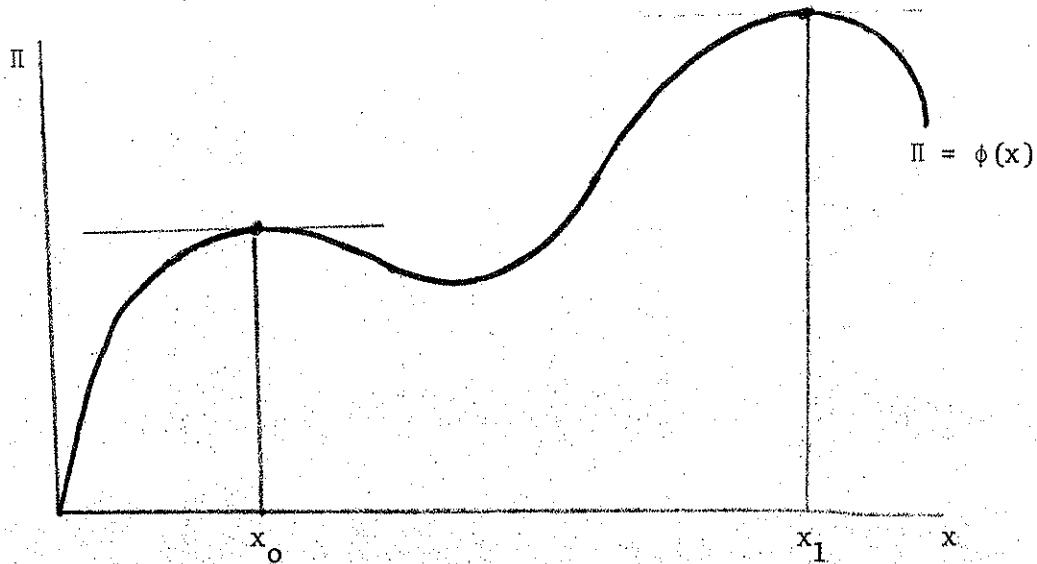
A Picture:

Again:

$$VMP = MRC$$

Note that:

TRC = Total Resource Cost = vx

MRC = Marginal Resource Cost = v

Local vs. Global Maxima

A problem that occurs with ill behaved functions.

Suppose a function:

two Local Maxima

        one at $x = x_0$

        one at $x = x_1$

when $x = x_0$ Local but not global

    $x = x_1$ Local and global

We assume our profit, cost, and productivity functions are well behaved (i.e., have only one optima) which is global and go on. Nothing really sacred about our well behaved functions.

Note also, the importance of the notion that are profit function has continuously turning tangents.

This ensures a unique solution to our profit maximization problem.
Otherwise:
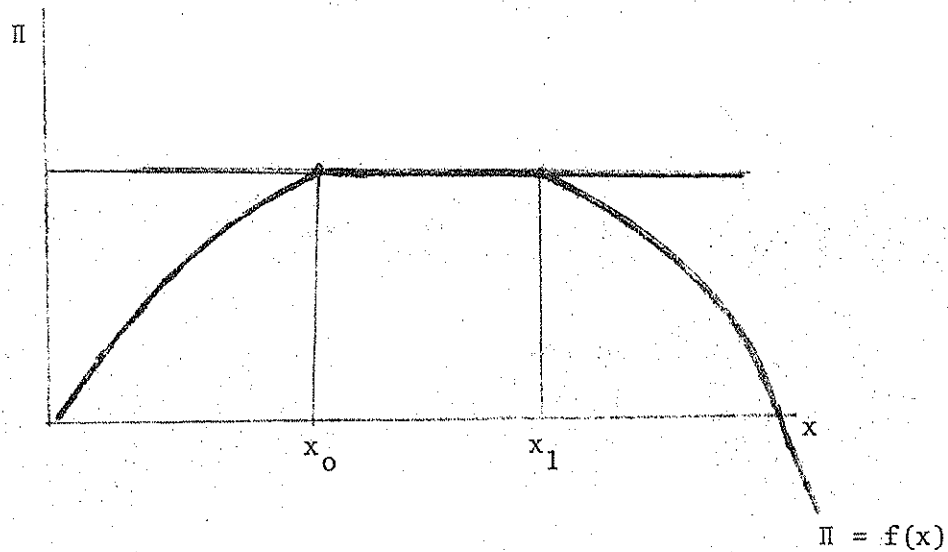
$$\Pi \text{ is maximized at } x_o, \text{ at } x_1, \text{ or anywhere in between.}$$

A more complex case assuming declining product prices. A simple application of the chain rule.

$$p = \phi (y)$$

$$\phi' < 0$$

(i.e., declining product prices with respect to output)

$$\Pi = \phi (y) y - vx$$

Note that:

$$p = \phi(y)$$
$$y = f(x)$$

A classic chain rule problem. Maximizing w.r.t. x

$$\frac{d\Pi}{dx} = y \frac{d\phi}{dy} \frac{dy}{dx} + p \frac{dy}{dx} = v$$

$\frac{d\phi}{dy}$ can be (crudely) interpreted as the change in product price associated with a change in the output level.

$$\frac{dy}{dx} = MPP$$

$$v = MRC$$

f.o.c.

$$VMP = MRC$$

A Numerical Example:

Suppose the production function is given by:

(1) $y = ax^{.7}$

what is

(a) MPP?

(b) APP?

(c) ϵ, the elasticity of production

if p = \$3
v = \$1
a = 1

(d) What value of x will maximize profits?

(e) How much y will be produced?

Exercise 3

1. What is the verbal interpretation of the phrase

$$\frac{d\Pi}{dx} = p \frac{dy}{dx} - v = 0$$

2. Why is the above equation termed a <u>necessary but not sufficient</u> condition?

3. What is meant by the term second order condition?

4. How does a local maximum differ from a global maximum?

5. Under what conditions

      (1) would it be impossible to find a finite solution to the problem of maximizing profits.

      (2) would it be possible to find several (perhaps an infinite number) of solutions to the problem of maximizing profits.

6. Suppose

      (1) a production function

$$y = x^{.4}$$

      price of output

$$p = p(y)$$

      price of input $= v$

Maximize profits.

What two fundamental rules of differentation are used to solve the above problem?

Introduction to Lagrangian Multipliers

Consider the following problem:

We wish to maximize

$$TR = py$$

where TR = total revenue

subject to the constraint (limitation) that:

$$C^o = vx$$

where

C^o = a constant equal to the total amount we
wish to spend in producing y.

Introduce the concept of a Lagrangian Multiplier:

The Lagrangian Multipler λ is some as yet undetermined number whose usefulness will become apparent. Actually, λ is a friend you've probably met before.

We have two unknowns, λ and x. p, v, and C^o are known constants (parameters). y is known for once x is known for we specify a unique rule for assigning unique values [y = f(x)].

Formulate the <u>Lagrangian expression</u>

(1) $\quad L = py + \lambda \ (C^o - vx = 0)$

Maximizing L with respect to x and λ
f.o.c.

(2) $\quad \dfrac{dL}{dx} = p \dfrac{dy}{dx} - \lambda v = 0$

(3) $\quad \dfrac{dL}{d\lambda} = C^o - vx = 0$

(2) and (3) are the two equations used in solving for our unknowns λ and x.

(4) $\quad p \dfrac{dy}{dx} = \lambda v$

$$\underset{\substack{\uparrow \\ \text{price of} \\ \text{product}}}{\text{MPP}} \cdot p = \lambda \cdot \underset{\substack{\text{price of} \\ \text{input}}}{v}$$

λ has a very special meaning--it is the "implicit worth" of an extra dollar spent in the purchase of x.

We are asking the question "Suppose I had an additional dollar to spend on x, how much more could I make with that dollar? In linear programming, λ is readily interpreted as the shadow price - the implicit worth of an extra unit of any resource. This is very similar to the concept of opportunity cost.

Note that, by (4)

$$\frac{p \dfrac{dy}{dx}}{v} = \lambda = \frac{MPP \cdot P}{v} = \frac{VMP}{v} = \lambda$$

if $\lambda = 1$, VMP $= v =$ MRC

If C^o were unlimited $\dfrac{VMP}{v} = 1$, or $\lambda = 1$, that is

the last dollar spent on x would contribute exactly 1 dollar to total revenue, and we would of course be at the point where VMP $= v =$ MRC.

If, in fact, C^o is an active constraint, that is C^o limits the amount of x that can be purchased λ will normally assume a value > 1, that is, the contribution of an additional dollar spent on x to total revenue would be greater than \$1. We would be at some point to the left of where MRC $=$ VMP Profits could be increased by further increasing purchases of x, but only by spending more, and C^o could no longer be held constant.

Consequently, profit maximization without constraints may very well make use of greater amounts of input x than if a total outlay constraint were inposed.

Note also that is C^o were a value large enough so that x could be used to beyond the point where $\dfrac{VMP}{v} = 1$, then, the constraint is not effective.

Exercise 4

1. What is the key difference between the solution to the profit
    maximization problem when

$$\Pi = py - vx$$

versus

$$L = py - \lambda(c^o - vx).$$

2. Under what conditions would the solutions to the above profit
    maximization problems be identical?

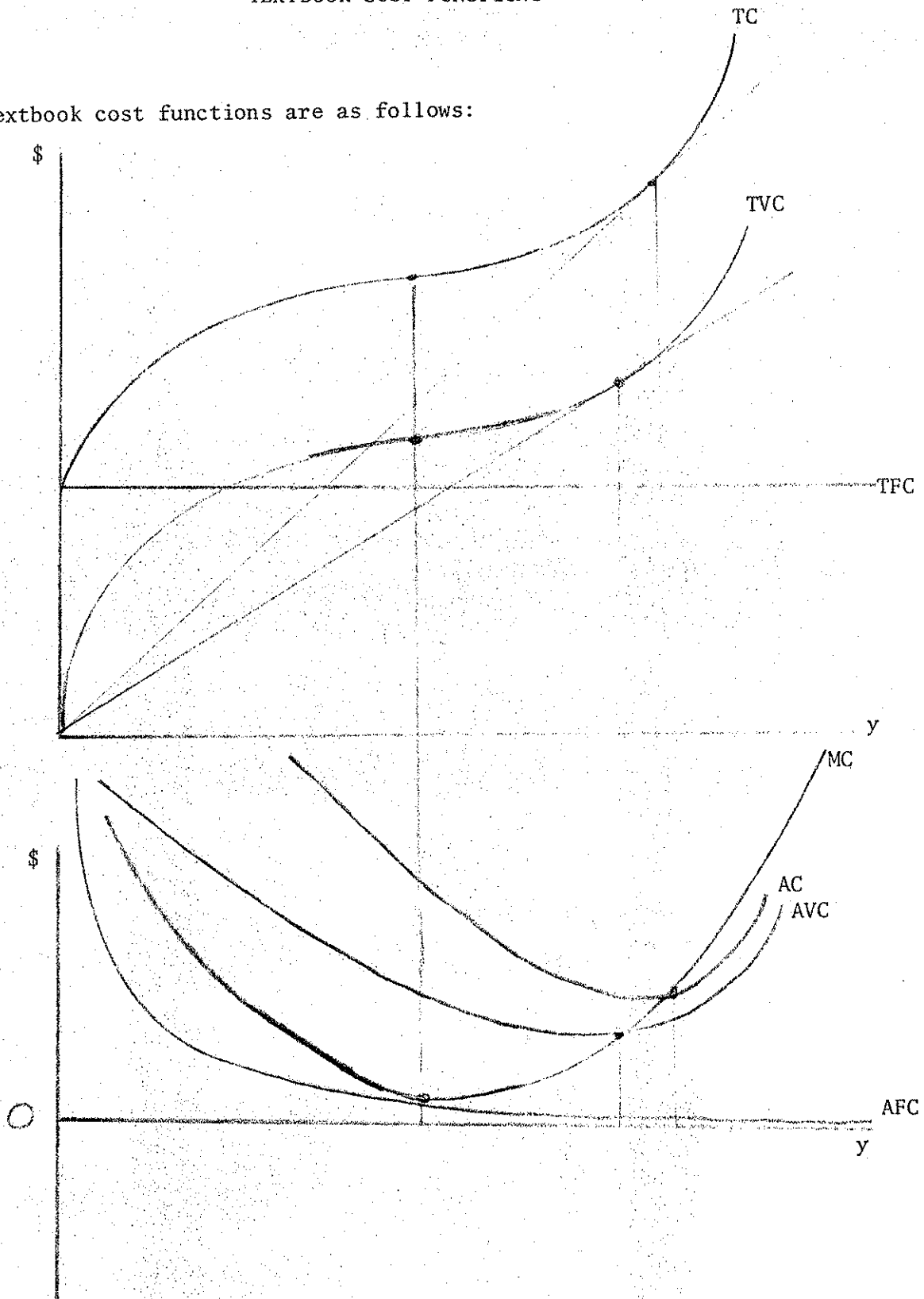3. Suppose a utility function is given by

$$u = u(z)$$

the Budget constraint is

$$I = pz$$

What is the marginal utility of money?

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CHAPTER IV

SIMPLE MATHEMATICS OF
TEXTBOOK COST FUNCTIONS

The textbook cost functions are as follows:

Average Fixed Cost

The equation:

$$AFC = \frac{TFC^*}{y}$$

(where: TFC^* = Total fixed costs assuming a constant value)

defines a rectangular hyperbola, the position in relation to the origin is dependent on the value of TFC^* that is assumed.
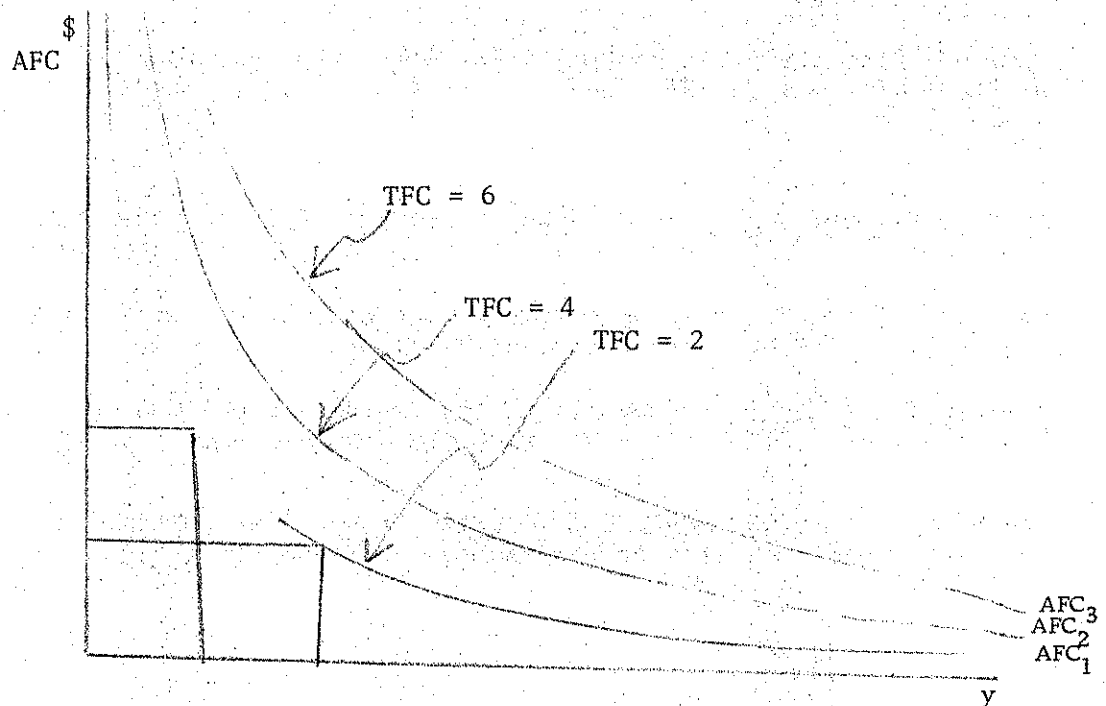
Note that $\dfrac{dAFC}{dy} = \dfrac{-TFC^*}{y^2} < 0$

AFC is a decreasing function with respect to output.

$$\frac{d^2AFC}{dy} = \frac{2TFC}{y^3} > 0$$

AFC is decreasing at a decreasing rate. (The rate of change of the slope is positive.)

Further note that a whole family of AFC curves can be drawn by merely assigning alternative values of TFC^*

Note also that the rectangle under the AFC curve is exactly equal to TFC. This is true because:

$$AFC = \frac{TFC}{y}$$

and

$$y \cdot AFC = TFC$$

Marginal Cost

As might be expected:

$$MC = \frac{dTC}{dy}$$

or

$$\int_0^{y_1} MCdy + TFC = TC$$

or

$$\int_0^{y_1} MCdy = TVC$$

Marginal cost assumes a minimum value where the inflexion point on the total cost function occurs. This is a single point of constant costs.

Average Variable Cost. AVC assumes a minimum value at the point where the ratio of TVC to y $\left(\frac{TVC}{y}\right)$ is minimum. This is equivalent to saying that AVC is minimum when a line passing out of the origin and intersecting TVC assumes a minimum slope.

Average Total Cost. Average Total Cost assumes a minimum value at the point where the ratio of TC to y $\left(\frac{TC}{y}\right)$ is minimum. This is equivalent to saying that AC is minimum when a line passing out of the origin and intersecting TC assumes a minimum slope.

Assuming TFC > 0, minimum AC is to the right of minimum AVC.

Marginal cost cuts AVC and AC at the respective minimum points. This should come as no shock, since a line passing out of the origin and intersecting TVC or TC at the respective minimum points is tangent to the TVC or TC curve and thus, also represents the slope (derivative) of the TC and TVC curves at the minimum points.

Simple Profit Maximization
From the Output Side

Define:

$$\Pi = TR - TC$$

to maximize Π

$$\frac{d\Pi}{dy} = \frac{dTR}{dy} - \frac{dTC}{dy} = 0$$

$$= MR - MC = 0$$

$$MC = MR$$

Mathematical Economics isn't as difficult as it seems!

Exercise 5

1. What does the shape of the Total Cost Function as shown in
    Figure 1 imply about the shape of the production function
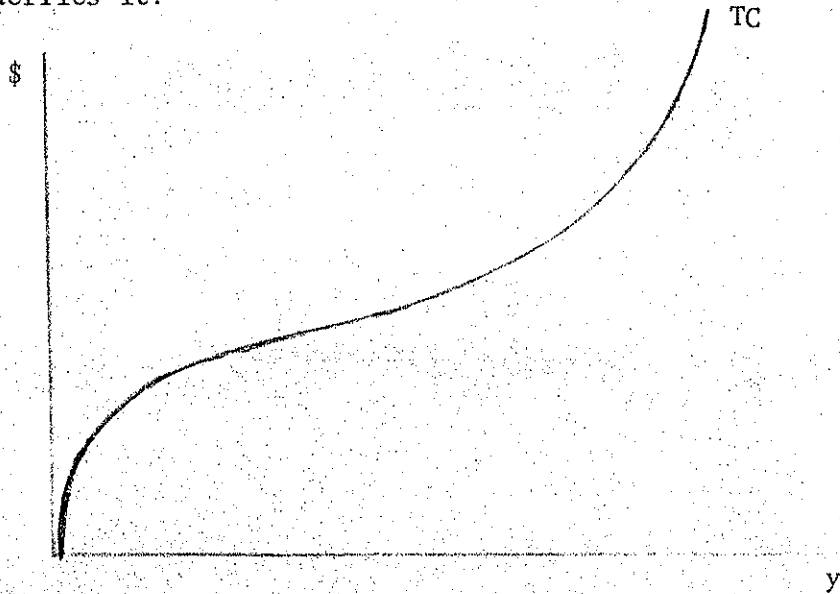   that underlies it?

Figure 1.

2. Why is it true that there is only a single point (output level)
    where MC = AC for the "textbook" production function?

3. Do average fixed costs ever reach zero?

4. Why is it true that MC assumes its minimum value at the inflexion point of the TC curve?

5. What happens to TC, TVC, AC, AVC and AFC in Stage III of production?

6. Suppose:

$$TC = g(y)$$

where:

TC = total cost
y = output

find:

MC

AC

Show that MC cuts AC at the minimum point of AC.

*Hint: Use the rule for differentiating fractions.

Exercise:

7. Show that:

MC lies below AC until AC reaches a minimum.

CHAPTER V

PRODUCTION WITH TWO
INPUTS AND A SINGLE OUTPUT

Consider a function

(1) $y = f(x_1, x_2 \mid x_3 \ldots x_n)$

or, simply

(2) $y = f(x_1, x_2)$

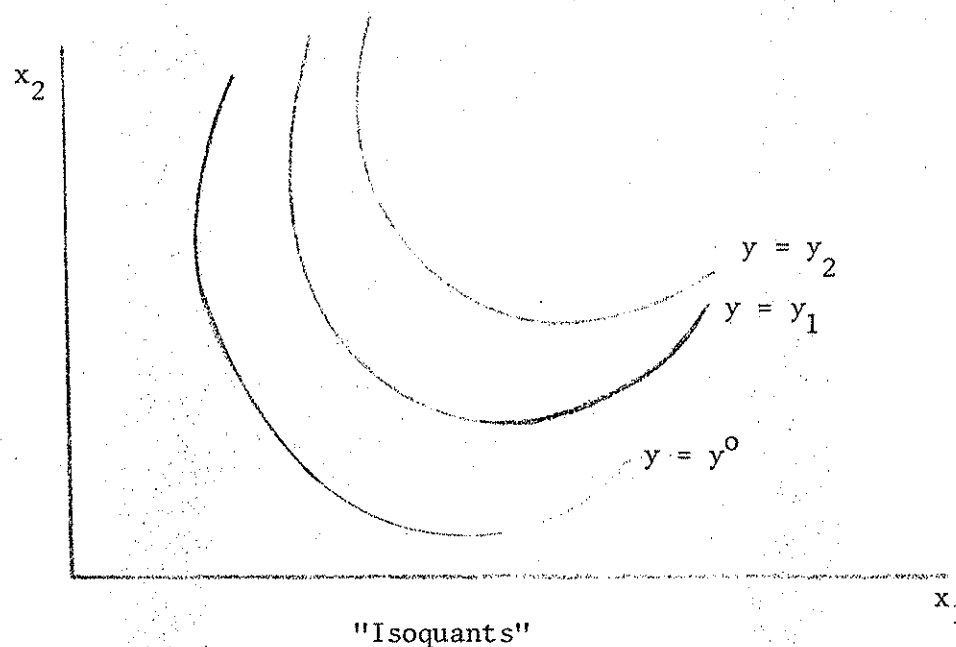Often the function is often also seen in its <u>implicit form</u>

(3) $0 = g(x_1, x_2, y)$

Suppose y is fixed at y^o

Then:

(4) $y^o = f(x_1, x_2)$

There are an infinite number of combinations of x_1 and x_2 which will

satisfy equation (4). The locus of these points is called an isoquant
(iso meaning equal; quant is short for quantity).

x_2

$y = y_2$

$y = y_1$

$y = y^o$

x_1

"Isoquants"

The slope of an isoquant

$$\frac{\Delta x_2}{\Delta x_1} \sim \frac{dx_2}{dx_1}$$

is often referred to as the <u>marginal rate of substitution</u> or <u>rate of technical substitution,</u>

MRS or RTS, and is a measure of <u>how well one input substitutes</u> for another. Note that if isoquants are drawn as depicted on preceding page, the RTS or slope is changing constantly along each curve.

Partial differentiation, total differentials and total derivatives.

We use the ∂ symbol to indicate derivatives with respect to one variable, holding the second variable constant.

i.e.,

$$\left(\frac{\partial y}{\partial x_1} \right) \quad x_2 = \text{constant}$$

$$\left(\frac{\partial y}{\partial x_2} \right) \quad x_1 = \text{constant}$$

A total differential can be crudely interpreted as the total change in a variable. For small changes in x_1 and x_2, a total differential quite closely approximates the total change.--Again the total differential really represents the change at a point. Starting with our original function

$$y = f(x_1, x_2)$$

We denote total change in finite increments as:

$$\Delta y = \underbrace{\frac{\partial f}{\partial x_1}}_{} \underbrace{\Delta x_1}_{\substack{\text{the change} \\ \text{in } x_1}} + \frac{\partial f}{\partial x_2} \Delta x_2$$

the change in output with respect to a change in the input x_1 a slope coefficient or parameter, not necessarily constant, but may vary along the function,

or, in less crude (limit) notation, the total differential is

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2$$

More importantly, $\frac{\partial f}{\partial x_1}$ is the marginal product of x_1, holding x_2 constant. Similarly, $\frac{\partial f}{\partial x_2}$ is the marginal product of x_2, holding x_1 constant.

Henderson and Quandt use the notation:

$$f_1 \text{ to denote } \frac{\partial f}{\partial x_1}$$

$$f_2 \text{ to denote } \frac{\partial f}{\partial x_2}$$

Along an isoquant, y is constant. Hence, along an isoquant, Δq and dq must also be constant. Hence, along an isoquant

(5) $\quad dy = 0 = \frac{\partial f}{\partial x_1} \, dx_1 + \frac{\partial f}{\partial x_2} \, dx_2$

rearranging (5)

$$-\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{dx_1}$$

The slope of the isoquant, the RTS or MRS is exactly equal to the negative ratio of the marginal productivities of each input. -- <u>This is a fundamental relationship</u> -- <u>one to which we will be returning again and again.</u>
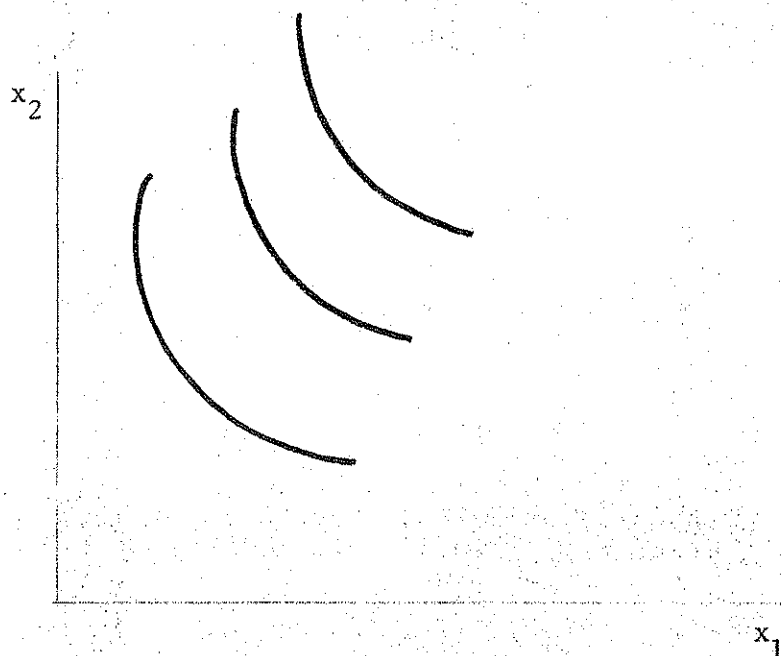
Some sample isoquant patterns:

Case I $\qquad \dfrac{dx_2}{dx_1} < 0, \qquad \dfrac{d^2 x_2}{d x_1^2} > 0$

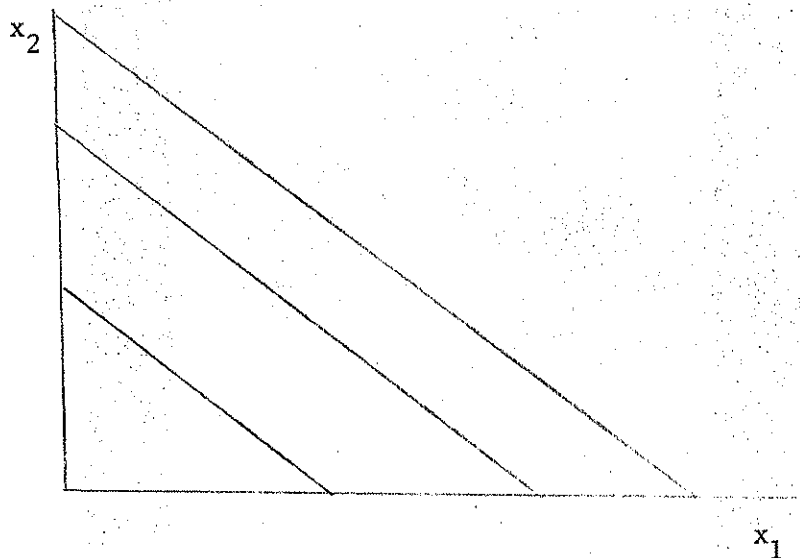slope of isoquant $\qquad$ rate of change of slope

Implies:

x_2

x_1

slope as negative and rate of change of slope is positive -- the classic case.

Case II.

$$\frac{dx_2}{dx_1} < 0 \qquad \frac{d^2x_2}{dx_1^2} = 0$$

Implies:
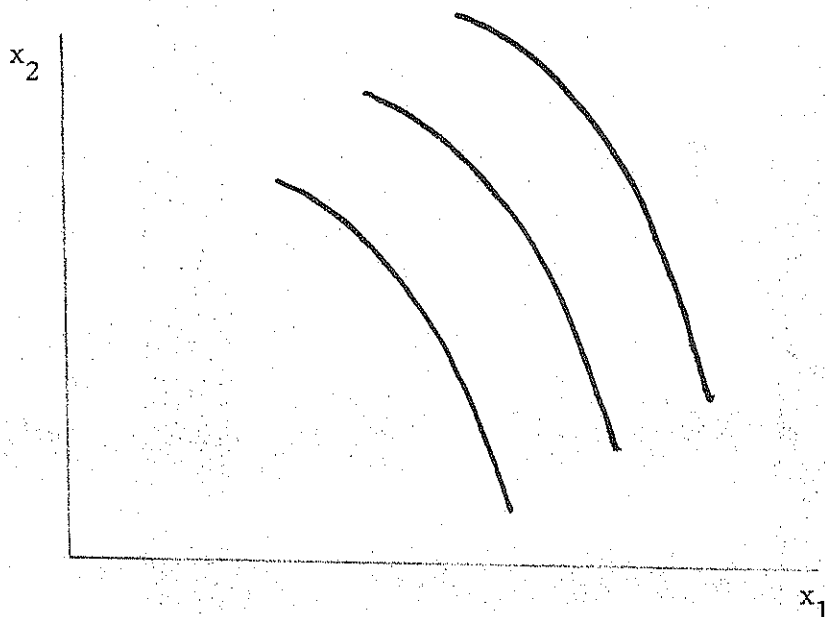
x_2

x_1

Case III.

$$\frac{dx_2}{dx_1} < 0, \quad \frac{d^2x_2}{dx_1^2} < 0$$
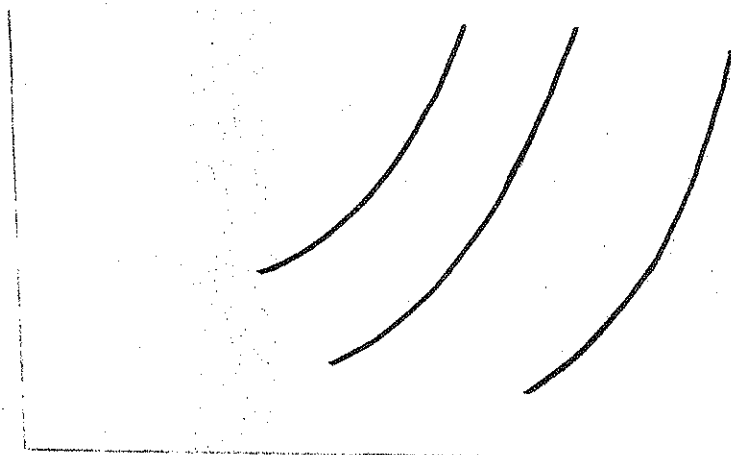
Isoquants are concave to the origin.

Implies:

Case IV.

$$\frac{dx_2}{dx_1} > 0 \quad \frac{d^2x_2}{dx_1^2} > 0$$

Isoquants Positive sloped, convex to SE.

Case V.

$$\frac{dx_2}{dx_1} > 0 \text{ and } \frac{d^2x_2}{dx_1^2} < 0$$

Isoquants Positive sloped, concave to SE.
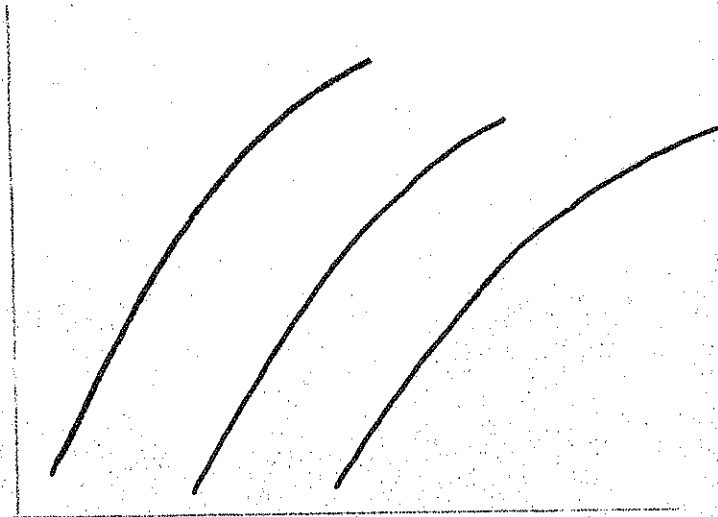
Case VI.

No factor substituteability.
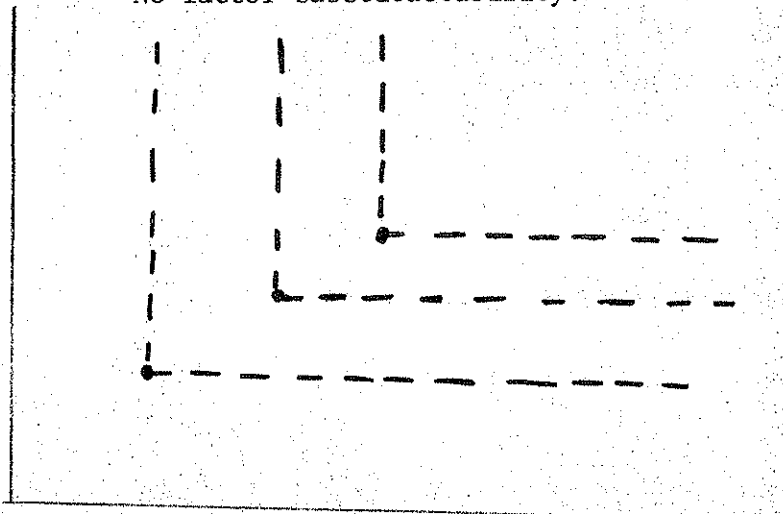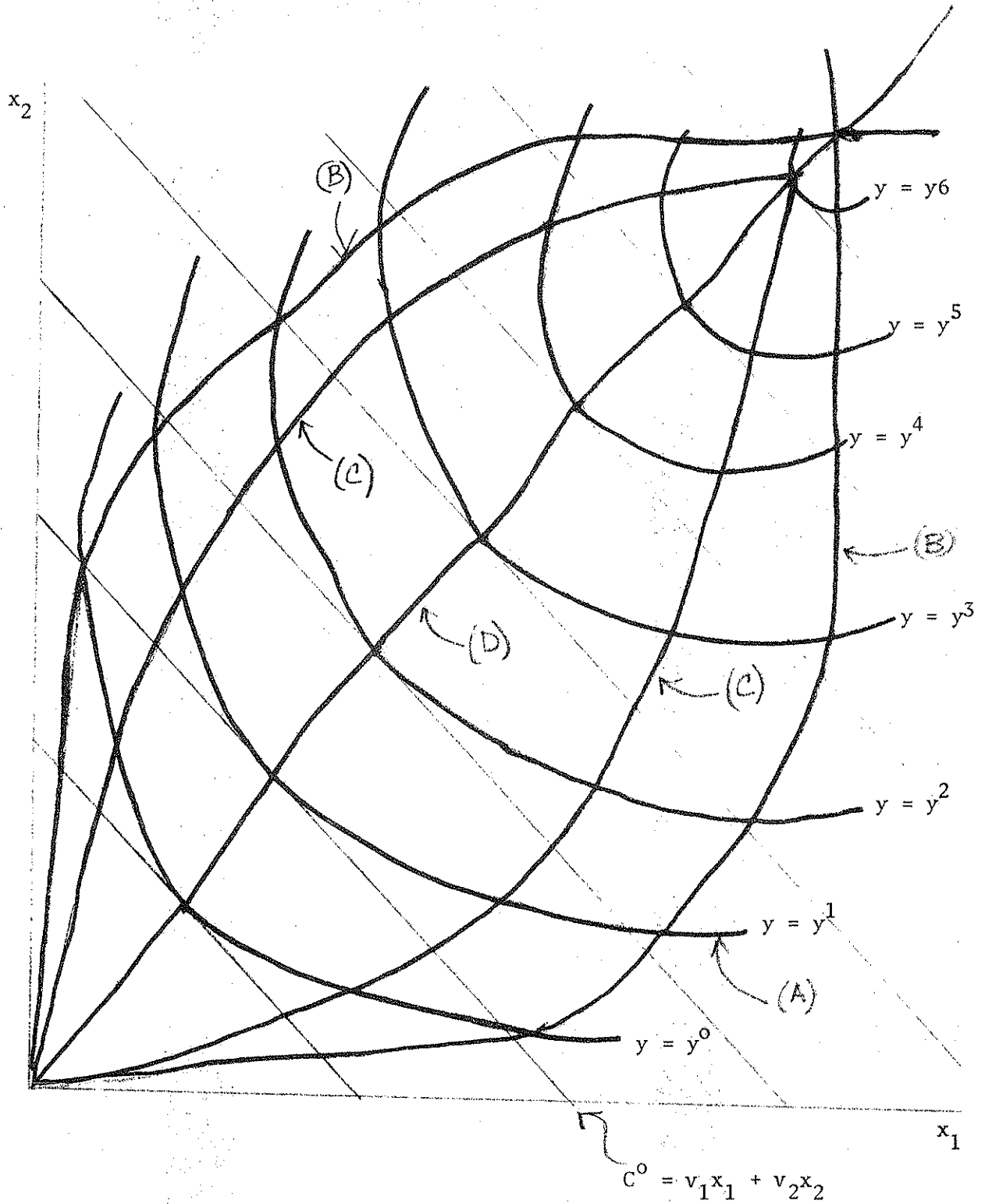
$$\frac{dx_2}{dx_1} \text{ not defined}$$

The textbook factor-factor model

x_2

(B)

(C)

(D)

(B)

(C)

(A)

$y = y6$

$y = y^5$

$y = y^4$

$y = y^3$

$y = y^2$

$y = y^1$

$y = y^o$

x_1

$c^o = v_1 x_1 + v_2 x_2$

Definitions:

    (a) <u>Isoquant</u> - a line of constant product - <u>usually</u> drawn <u>convex to the origin</u> and with <u>continuously turning tangents</u>. Neither condition is <u>necessary</u>.

    (b) <u>Ridge Line</u> - lines connecting points of zero or infinite slope on isoquants.

$$\text{i.e.,} \quad \frac{dx_2}{dx_1} = 0 \quad \text{or} \quad \frac{dx_2}{dx_1} = +\infty$$

        alternately

$$\frac{dx_1}{dx_2} = +\infty \quad \text{or} \quad \frac{dx_1}{dx_2} = 0$$

Note that the area internal to the ridge line is an area which lies in Stage II for both x_1 and x_2. (WHY?) The area NE of the right ridge line lies in Stage III for x_1 and Stage I for x_2. (WHY?) The area to the NW of the left ridge line lies in Stage III for x_2 and Stage I for x_1. (WHY?) Ridge lines cross each other at the point of maximum total product. (WHY?)

Exercise:

    Suppose a production function of the form (Cobb Douglas)

$$y = Ax_1^{\alpha_1} x_2^{1-\alpha_1}$$

    where do the ridge lines lie?

<u>Isocline</u> - any line connecting points on isoquants of equal slope. Connects points where:

$$\frac{dx_i}{dx_j} = k$$

Where k is some constant.

    (c) <u>Pseudo-Scale Line</u> - A specialized isocline connecting points on isoquants where

$$p \frac{\partial f}{\partial x_1} = v_1$$

$$VMP_{x_1} = v_1$$

        for alternative values of x_2

$$p \frac{\partial f}{\partial x_2} = v_2$$

$$\text{VMP}_{x_2} = v_2 \text{ for alternative values of } x_1$$

where:

v_1 and v_2 are the respective prices of the inputs x_1 and x_2, p = the price of output.

Pseudo scale lines enclose the area of rational production.

Pseudo scale lines converge at the point of maximum profit.'

Note that the point where the pseudo scale lines converge.

$$\frac{p \frac{\partial f}{\partial x_1}}{v_1} = \frac{p \frac{\partial f}{\partial x_2}}{v_2}$$

$$\frac{\text{VMP}_{x_1}}{v_1} = \frac{\text{VMP}_{x_2}}{v_2}$$

Further:

$$\frac{\text{VMP}_{x_1}}{v_1} = \frac{\text{VMP}_{x_2}}{v_2} = \ldots \frac{\text{VMP}_{x_n}}{v_n} = 1$$

The marginal value product of each factor divided by the respective factor prices are equal for all inputs, and in the unconstrained case, equal to 1.

If production is constrained by the availability of dollars for the pruchase of inputs,

$$\frac{\text{VMP}_{x_i}}{v_i} > 1$$

Note that if

$$p \frac{\partial f}{\partial x_1} = v_1$$

$$p \frac{\partial f}{\partial x_2} = v_2$$

if $v_2 = 1$ (a unit price)　then　$\dfrac{dx_2}{dx_1} = v_1$

further:

$$\text{if}\quad p \frac{\partial f}{\partial x_1} = v_1$$

$$\frac{p \dfrac{\partial f}{\partial x_1}}{v_1} = 1$$

Expansion path.

Suppose a constraint of the form:

(1)　$C^o = g(x_1,\ x_2;\ v_1,\ v_2)$

More specifically, suppose the constraint to be represented by an isooutlay curve or isocost curve

(2)　$C^o = v_1 x_1 + v_2 x_2$

where

C^o = a constant equal to some arbitiary outlay for x_1 and x_2.

v_1 and v_2 = prices of x_1 and x_2

Note that:

Our constraint function is really a linear combination.

v_1 and v_2 are prices which act as weights on the relative contribution of x_1 and x_2 to total outlay.

Note that along the horizontal axis, $x_1 = C^o/v_1$, and along the vertical axis $x_2 = C^o/v_2$.   (WHY?)

Taking the total differential of (2) with respect to x_1 and x_2 (assume v_1 and v_2 to be constant parameters) yields (3) $dC^o = v_1 dx_1 + v_2 dx_2$, but $dC^o = 0$ since C^o is a constant.

Hence:

$$0 = v_1 dx_1 + v_2 dx_2$$

or

$$\frac{v_1}{v_2} = \frac{dx_2}{dx_1}$$

Further, in finite increments

$$\frac{v_1}{v_2} = \frac{\Delta x_2}{\Delta x_1}$$

The slope of our constraint function is equal to the negative inverse ratio of input prices. Note that the distance of the constraint function from the origin ($x_1=0$, $x_2=0$) on the graph is determined by the value of C^o. The slope is <u>always</u> equal to the constant $\left(\frac{v_1}{v_2}\right)$ <u>a ratio of weights.</u>

A line connecting all points on a set of isoquants where the slope of the isoquants are equal to $\left(\frac{v_1}{v_2}\right)$ is known as an <u>Expansion Path</u>.

An Expansion Path is usually shown in intermediate level texts to be a straight line passing through the origin. Actually, expansion path need not be a straight line. <u>Homothetic</u> production functions generate linear expansion paths.

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Introduction to Maximization:
the two input case

Suppose a function

$$(1) \quad y = f(x_1, x_2)$$

(1) reaches an <u>unconstrained maximum</u> (or perhaps minimum) at the point where the <u>following first order conditions</u> (f.o.c.) are met

$$\frac{\partial y}{\partial x_1} = \frac{\partial f}{\partial x_1} = 0$$

$$\frac{\partial y}{\partial x_2} = \frac{\partial f}{\partial x_2} = 0$$

at the point where the marginal product function equals zero for both inputs

For a maximum, second order conditions (sufficient conditions) are met if, (1)

$$\frac{\partial^2 f}{\partial x_1^2} < 0$$

and

$$(2) \quad \frac{\partial^2 f}{\partial x_1^2} \cdot \frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial x_1 x_2} \cdot \frac{\partial^2 f}{\partial x_1 x_2} > 0$$

The <u>quadratic form</u> is <u>negative definite.</u> Alternately, in matrix notation

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 x_2} \\[4mm] \dfrac{\partial^2 f}{\partial x_2 x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{vmatrix} > 0$$

$$\text{and} \quad \begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} \end{vmatrix} < 0$$

The principle minors of the relevant Hessian determinant must alternate in sign for a maximum. Study H & Q, pp. 401-402.

These conditions ensure that the production function is concave. The strict inequalities hold if the function has continuously turning tangents.

Isoquants derived from a strictly concave function are convex to the origin.

Two production functions are needed to derive sets of isoquants.

Fix x_2 at x_2^*

x_2

$x_2 = x_2^*$

y_3 y_2

y_1

$y = y_0$

$\left(x_1 \mid x_2 = x_2^* \right)$

y

y_3

y_2

y_1

y_0

TPP

X_1
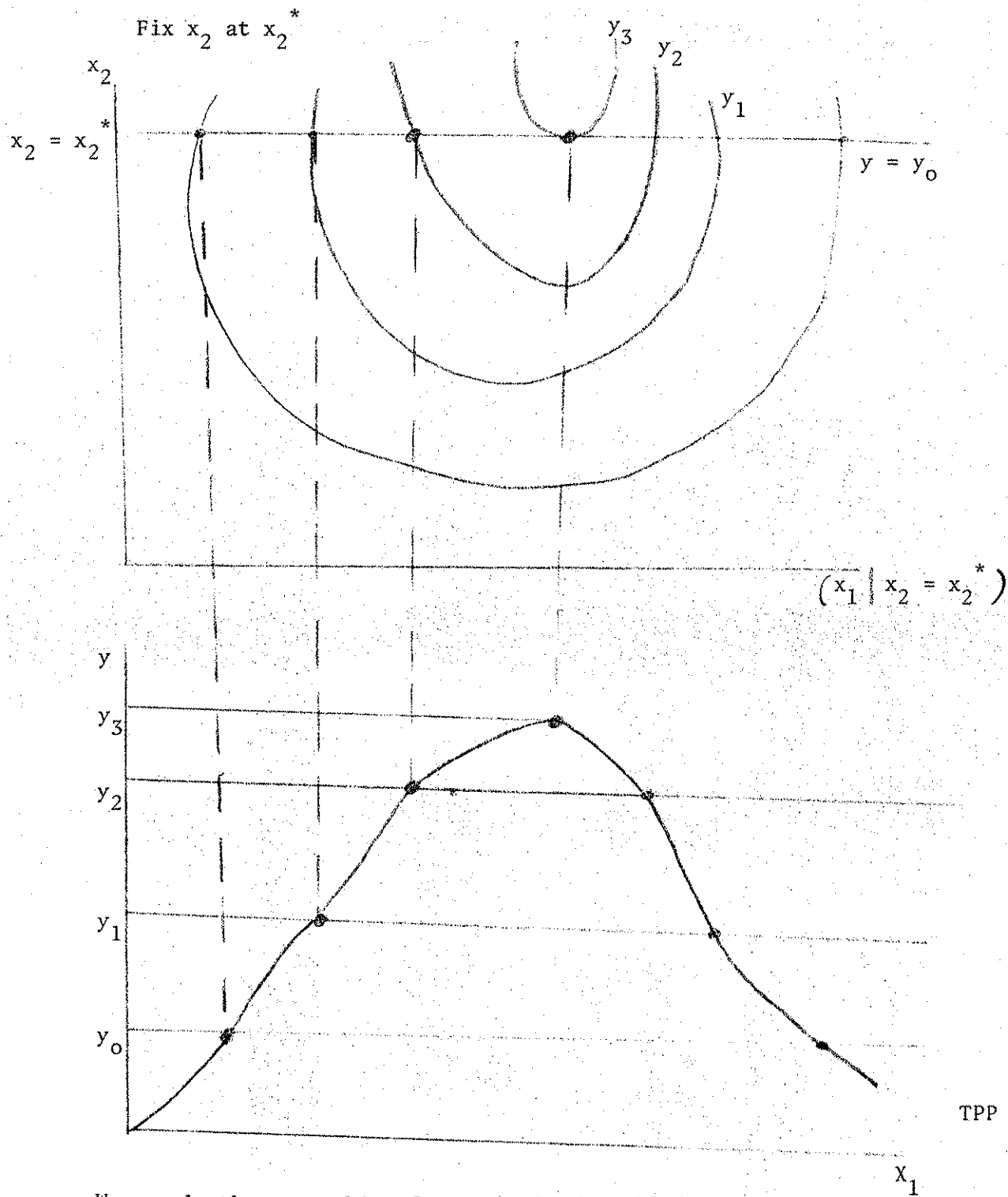
We can do the same thing for x_2 by holding x_1 fixed. Hence:
two production functions underlie every set of isoquants.

Profit maximization, the 2 input case.

Suppose a production function:

$$y = f(x_1, x_2)$$

A cost function:

$$C = g(x_1, x_2; v_1, v_2)$$

A profit function:

$$\Pi = py - C$$

$$\Pi = pf(x_1, x_2) - g(x_1, x_2; v_1, v_2)$$

Maximum profits (or perhaps minimum profits) unconstrained:

$$\frac{\partial \Pi}{\partial x_1} = p\frac{\partial f}{\partial x_1} - \frac{\partial g}{\partial x_1} = 0$$

$$\frac{\partial \Pi}{\partial x_2} = p\frac{\partial f}{\partial x_1} - \frac{\partial g}{\partial x_2} = 0$$

or

$$p\frac{\partial f}{\partial x_1} = \frac{\partial g}{\partial x_1}$$

$$p\frac{\partial f}{\partial x_2} = \frac{\partial g}{\partial x_2}$$

$\frac{\partial f}{\partial x_i}$ = marginal product function for the ith input.

$\frac{\partial g}{\partial x_i}$ = change in total outlay with respect to a change in the use of the ith input.

The Marginal Resource Cost Function.

Suppose the specific form of g to be the linear combination

$$g(x_1, x_2) = C = v_1 x_1 + v_2 x_2$$

where

$$v_i = \text{the respective input prices,}$$

then

$$\frac{\partial g}{\partial x_i} = v_i = \text{MRC} = \text{price of the input.}$$

Further

$$\frac{\dfrac{\partial f}{\partial x_1}}{\dfrac{\partial f}{\partial x_2}} = \frac{v_1}{v_2}$$

and

$$\frac{dx_2}{dx_1} = \frac{v_1}{v_2}$$

$$\text{RTS} = \text{inverse input price ratio}$$

defines a point where pseudo scale lines intersect the point of maximum profit.

Langrangian Multipliers - Constrained Output Maximization

Suppose C to be constrained at some value C^o, i.e., no more than $\$C^o$ can be spent on x_1 and x_2.

Output maximization subject to a cost constraint. Formulate the Lagrangian expression

$$L = f(x_1,\ x_2) - \lambda(C^o - v_1 x_1 - v_2 x_2)$$

f.o.c.

$$\frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda v_1 = 0$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda v_2 = 0$$

$$\frac{\partial L}{\partial \lambda} = C^o - v_1 x_1 - v_2 x_2 = 0$$

Further:

$$\frac{\partial f}{\partial x_1} = \lambda \, v_1$$

$$\frac{\partial f}{\partial x_2} = \lambda \, v_2$$

Hence: RTS = inverse price ratio

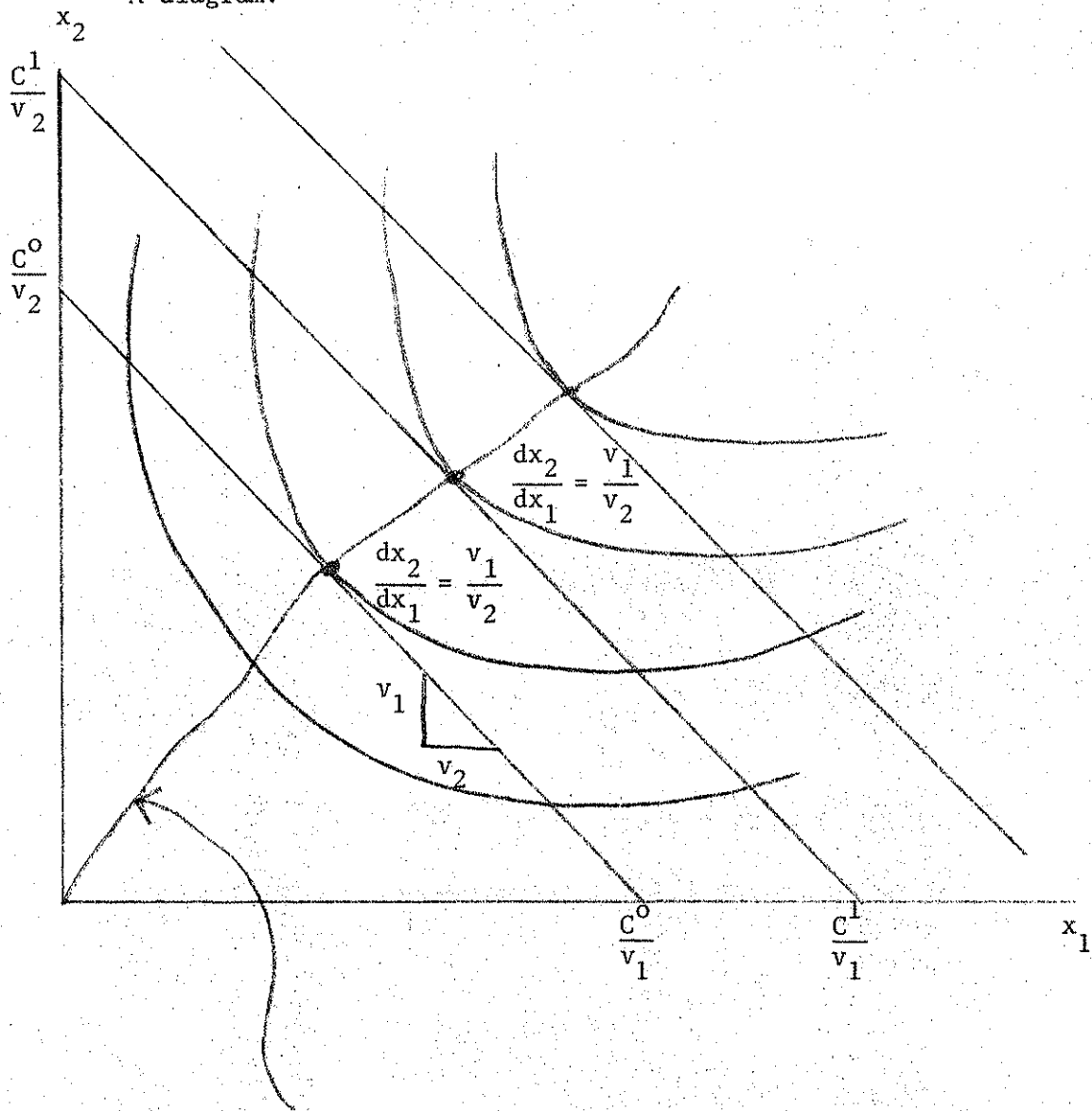$$\frac{dx_2}{dx_1} = \frac{v_1}{v_2}$$

for any $C = C^o$

Refers to point of tangency between isooutlay curve and isoquant. Represents maximum profit for any arbitrary outlay, C^o.

Note also:

$$\frac{\frac{\partial f}{\partial x_1}}{v_1} = \frac{\frac{\partial f}{\partial x_2}}{v_2} = \lambda$$

λ is the implicit value of the relaxation of the cost constraint by 1 dollar.

A diagram:

$$\frac{dx_2}{dx_1} = \frac{v_1}{v_2}$$

$$\frac{dx_2}{dx_1} = \frac{v_1}{v_2}$$

Expansion Path

Least Cost Combination of Inputs

Note that the f.o.c. for the least cost combination of inputs
is defined as the condition where

$$\frac{p \cdot MPP_{x_1}}{v_1} = \frac{p \cdot MPP_{x_2}}{v_2} = \lambda$$

This condition defines an infinity of points along the expansion
path. Notice that:

> "for every value of C that one assumes, there
> corresponds a unique least cost combination,
> there is in fact an infinity of points of
> least cost combination. There is, however,
> only one global point of profit maximization
> where λ assumes a value of 1 and the pseudo
> scale lines intersect."

s.o.c. for maximum

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 x_2} & - v_1 \\[2em] \dfrac{\partial^2 f}{\partial x_2 x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & - v_2 \\[2em] -v_1 & -v_2 & 0 \end{vmatrix} > 0$$

denote

$$\frac{\partial^2 f}{\partial x_i x_j} \quad as \quad f_{ij}$$

then

$$\begin{vmatrix} f_{11} & f_{12} & - v_1 \\[1em] f_{21} & f_{22} & -v_2 \\[1em] -v_1 & -v_2 & 0 \end{vmatrix} > 0$$

or

$$f_{11} \cdot f_{22} \cdot 0 + (f_{12})(-v_2)(-v_1)$$

$$+ (-v_1)(f_{21})(-v_2)$$

$$-[(-v_1)(-v_1)(f_{22}) + f_{11}(-v_2)(-v_2)$$

$$+ (f_{12})(f_{21})(0)]$$

or

$$f_{12}v_2v_1 + f_{21}v_1v_2$$

$$- f_{22}v_1^2 - f_{11}v_2^2$$

but:

$$f_{12} = f_{21} \qquad \text{Why?}$$

then:

$$- f_{22}v_1^2 - f_{11}v_2^2 + 2f_{12}v_2v_1 > 0$$

but:

$$v_i = \frac{f_i}{\lambda}$$

Hence:

$$- f_{22}\frac{f_1^2}{\lambda^2} - f_{11}\frac{f_2^2}{\lambda^2} + 2f_{12}\frac{f_2}{\lambda}\frac{f_1}{\lambda} > 0$$
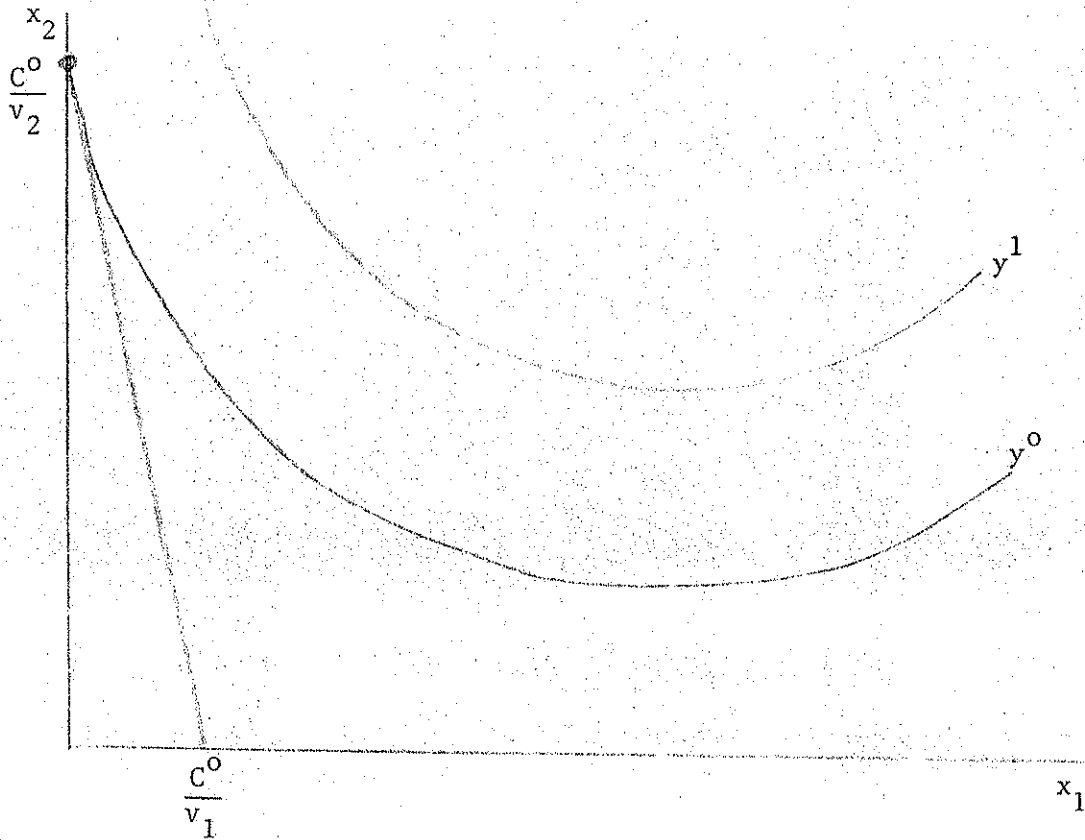
Further:

$$2f_{12}f_1f_2 - f_{11}f_2^2 - f_{22}f_1^2 > 0$$

See H and Q, equation 2-10 and 2-15, pp. 15 and 19.

This is a sufficient condition for a global maximum. By advanced methods, it has been proved that the function $f(x_1, x_2)$ (or, for that matter $f(x_1, x_2, \ldots, x_n)$ cannot have more than one constrained maximum over an interval if the above determinantal condition holds over the interval. See H and Q, pg. 406.

Further Remarks

***These first and second order conditions hold if and only if values of x_1 and x_2 are held at strictly positive levels. In other words we do not allow the situation where

Derivatives are defined only on open sets.

Need Kukn - Tucker conditions to show what is pictured above mathematically.

Generalization to n variables

Of course:

f.o.c.

$$\frac{\partial f}{\partial x_i} = \lambda \, v_i$$

$$\frac{\partial f}{\partial x_j} = \lambda \, v_j \quad \text{for all } i, j = 1, \ldots n$$

Hence:

$$\frac{MPP_{x_1}}{v_1} = \frac{MPP_{x_2}}{v_2} = \ldots = \frac{MPP_{x_n}}{v_n} = \lambda$$

Further:

s.o.c.

$$(-1)^n \begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} & -v_1 \\ f_{21} & f_{22} & \cdots & f_{2n} & -v_2 \\ \vdots & & & \vdots & \\ f_{n1} & f_{n2} & \cdots & f_{nn} & -v_n \\ -v_1 & -v_2 & & -v_n & 0 \end{vmatrix} > 0$$

Where n is the number of inputs.

For a maximum this is the negative definite quadratic form.

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Exercise 6

1. Suppose a production function is given by:

$$y = Ax_1^{.3} \, x_2^{.6}$$

What is the total differential dy?

2. Calculate the RTS for the above production function if y = some constant y^*.

3. Draw the isoquants that would be associated with the function in (1).

4. Find the location of the ridge lines for isoquants generated by the production function (1).

5. What is implied by the condition?

$$\frac{VMP_{x_1}}{v_1} > \frac{VMP_{x_2}}{v_2}$$

6. What is implied by the condition?

$$\frac{VMP_{x_1}}{v_1} = \frac{VMP_{x_2}}{v_2} = 3$$

7. A "pseudo scale line" is defined as a specialized isocline connecting points on isoquants where

$$p \frac{\partial f}{\partial x_1} = v_1$$

and

$$p \frac{\partial f}{\partial x_2} = v_2$$

Why is it true that pseudo scale lines cross at the point where $\lambda = 1$ and this is the unconstrained point of maximum profit?

8. Under what conditions is it possible to write the implicit function

$$g(y, x_1, x_2) = 0$$

in the form:

$$y = f(x_1, x_2)$$

Hint: Check a calculus book on the implicit function theorem.

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Elasticities of Substitution

The elasticity of substitution is defined as a measure of the
rate at which substitution between two inputs takes place.--
Proportionate rate of change in the input ratio divided by the
proportionate rate of change in the RTS. More formally:

$$\sigma \equiv \frac{\dfrac{\dfrac{\partial y}{\partial x_1}}{\dfrac{\partial y}{\partial x_2}}}{\dfrac{x_2}{x_1}} \cdot \frac{d\left(\dfrac{x_2}{x_1}\right)}{d\left(\dfrac{\dfrac{\partial y}{\partial x_1}}{\dfrac{\partial y}{\partial x_2}}\right)}$$

Note the difference between the formula for the elasticity of
substitution and the elasticity of production.

For a production function with convex isoquants, both the slope
of the indifference curve, $\dfrac{\Delta x_2}{\Delta x_1}$ and the ratio of total use of the

2 inputs $\left(\dfrac{x_2}{x_1}\right)$ is declining as x_1 is substituted on x_2.

The elasticity of substitution tells us whether or not these ratios
are declining proportionately. So long as $\dfrac{\partial y}{\partial x_1}$ (MPP_1) and

$\dfrac{\partial y}{\partial x_2}$ (MPP_2) are positive, the elasticity of substitution is always

positive. See proof, H and Q, pg. 62.

Suppose:

$$y = Ax_1^{\alpha_1} \, x_2^{\alpha_2}$$

then

$$\frac{\partial y}{\partial x_1} = \alpha_1 \, Ax_1^{\alpha_1-1} \, x_2^{\alpha_2}$$

$$\frac{\partial y}{\partial x_2} = \alpha_1 \, Ax_1^{\alpha_1} \, x_2^{\alpha_2-1}$$

$$\frac{\dfrac{\partial y}{\partial x_1}}{\dfrac{\partial y}{\partial x_2}} = \frac{\alpha_1}{\alpha_2} \, \frac{x_2}{x_1}$$

So:

$$\sigma = \frac{\dfrac{\alpha_1}{\alpha_2} \, \dfrac{x_2}{x_1}}{\dfrac{x_2}{x_1}} \quad \frac{d\left(\dfrac{x_2}{x_1}\right)}{d\left(\dfrac{\alpha_1}{\alpha_2} \, \dfrac{x_2}{x_1}\right)}$$

but:

$$d\left(\frac{\alpha_1}{\alpha_2} \quad \frac{x_2}{x_1}\right) = \frac{\alpha_1}{\alpha_2} \quad d\left(\frac{x_2}{x_1}\right)$$
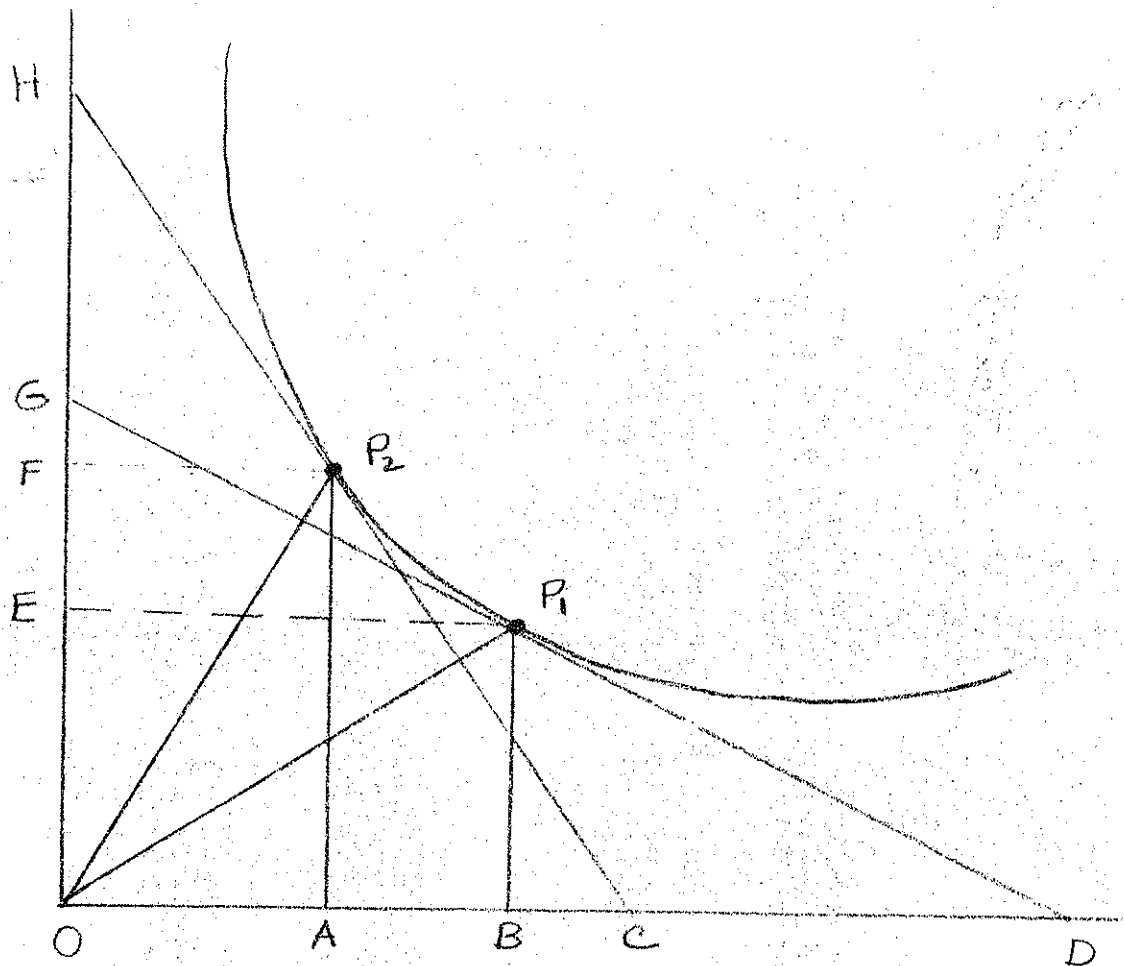
so:

$$= \frac{\dfrac{\alpha_1}{\alpha_2} \, \dfrac{x_2}{x_1}}{\dfrac{x_2}{x_1}} \quad \frac{d\left(\dfrac{x_2}{x_1}\right)}{\dfrac{\alpha_1}{\alpha_2} \, d\left(\dfrac{x_2}{x_1}\right)}$$

$$= 1$$

All Cobb Douglas type functions have constant unitary elasticities of substitution even if $\sum\limits_{i=1}^{n} \alpha_i \neq 1$.

A Diagramatic Illustration of the
Concept of an Elasticity of Substitution

The elasticity of substitution is equal to:

$$\frac{\dfrac{OF}{OA} - \dfrac{OE}{OB}}{\dfrac{OH}{OC} - \dfrac{OG}{OD}}$$

Over the finite range from P_1 to P_2

Exercise 7

1. Suppose that a tenant receives only r percent of the revenue from the sale of crops on a farm being rented (0<r<100). How, in the eyes of the tenant, are the first order conditions for profit maximization altered with respect to the purchase of variable inputs x_1 and x_2 (i.e., fertilizer and seed)?

2. Suppose that the tenant and landlord agree to share expenses-- the tenant to pay s percent of the cost of seed and fertilizer, the landlord to pay 100-s percent. How are first order conditions for profit maximization altered?

3. Under what conditions would the purchase of x_1 and x_2 by the tenant be the same as if the tenant owned the land?

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Constrained Cost Minimization Subject to an Output Constraint.

Suppose we wish to minimize our (linear combination) cost function subject to the constraint that

$$y^o = f(x_1 x_2)$$

where output y is fixed at some arbitrary level y^o. Form the lagrangian:

$$L = v_1 x_1 + v_2 x_2 + \mu [y^o - f(x_1, x_2)]$$

$$\frac{\partial L}{\partial x_1} = v_1 - \mu \frac{\partial f}{\partial x_1} = 0$$

$$\frac{\partial L}{\partial x_2} = v_2 - \mu \frac{\partial f}{\partial x_2} = 0$$

$$\frac{\partial L}{\partial \mu} = y^o - f(x_1, x_2) = 0$$

or

$$RTS = \frac{dx_1}{dx_2} = \frac{v_2}{v_1}$$

$$\mu = \frac{1}{\lambda}$$

s.o.c.

$$\begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -f_1 \\ -\mu f_{12} & -\mu f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} < 0$$

Negative, opposite of that for maximization. How derived ?
Starting with f.o.c. :

(1) $$v_1 - \mu \frac{\partial f}{\partial x_1} = 0$$

(2) $\qquad v_2 - \mu \dfrac{\partial f}{\partial x_2} = 0$

(3) $\qquad y^o - f(x_1, x_2) = 0$

$$\frac{\partial (1)}{\partial x_1} = -\mu \frac{\partial^2 f}{\partial x_1^2} = 0$$

$$\frac{\partial (1)}{\partial x_2} = -\mu \frac{\partial^2 f}{\partial x_1 x_2} = 0$$

$$\frac{\partial (1)}{\partial \mu} = -\frac{\partial f}{\partial x_1} = 0$$

$$\frac{\partial (2)}{\partial x_1} = -\mu \frac{\partial^2 f}{\partial x_2 x_1} = 0$$

$$\frac{\partial (2)}{\partial x_2} = -\mu \frac{\partial^2 f}{\partial x_2^2} = 0$$

$$\frac{\partial (2)}{\partial \mu} = \frac{\partial f}{\partial x_2} = 0$$

$$\frac{\partial (3)}{\partial x_1} = \frac{\partial f}{\partial x_1} = 0$$

$$\frac{\partial (3)}{\partial x_2} = \frac{\partial f}{\partial x_2} = 0$$

$$\frac{\partial (3)}{\partial \mu} = 0$$

form the matrix

$$\begin{vmatrix} -\mu f_{11} & -\mu f_{12} & -f_1 \\ -\mu f_{21} & -\mu f_{22} & -f_2 \\ -f_1 & -f_2 & 0 \end{vmatrix} < 0$$

See proof in H and Q, pg. 66 that the preceding determinant is equivalent to

$$\begin{vmatrix} f_{11} & f_{12} & -v_1 \\ f_{21} & f_{22} & -v_2 \\ -v_1 & -v_2 & 0 \end{vmatrix} > 0$$

Almost like a primal and dual problem in LP.

The solution that maximizes output subject to a cost constraint is the same solution that minimizes cost subject to an output constraint.

Partial elasticities of production

Earlier, we defined ε as $\dfrac{\%\Delta \text{ in output}}{\%\Delta \text{ in input}}$ for the single input case

$$\varepsilon = \frac{\dfrac{dy}{y}}{\dfrac{dx}{x}} = \frac{dy}{dx} \frac{x}{y}$$

For the 2 input case

$$\varepsilon^1 = \left(\frac{\dfrac{\partial y}{y}}{\dfrac{\partial x_1}{x_1}} \right)_{x_2 = \text{const.}}$$

$$= \left(\frac{\partial y}{\partial x_1} \cdot \frac{x_1}{y} \right)_{x_2 = \text{const.}}$$

$$\varepsilon^2 = \left(\frac{\dfrac{\partial y}{y}}{\dfrac{\partial x_2}{x_2}} \right)_{x_1 = \text{const.}}$$

$$= \left(\frac{\partial y}{\partial x_2} \cdot \frac{x_2}{y} \right)_{x_2 = \text{const.}}$$

In general, in the n variable case

$$\epsilon^j = \left(\frac{\frac{\partial y}{y}}{\frac{\partial x_j}{x_j}} \right) \qquad \text{for all} \quad x_i = 1, \text{ n}, \text{ i} \neq j \ = \text{const.}$$

$$= \left(\frac{\partial y}{\partial x_j} \ \frac{x_j}{y} \right) \qquad \text{for all} \quad x_i = 1, \text{ n}, \text{ i} \neq j \ = \text{const.}$$

Hence:

$$\epsilon^j = \left(\frac{MPP_j}{APP_j} \right) \qquad \begin{array}{l} \text{for all} \\ \text{i} = 1, \text{ n}, \text{ i} \neq j \ = \text{const.} \end{array}$$

The function coefficient:

$$E = \sum_{i=1}^{n} \epsilon^i$$

or

$$E = \frac{MPP_1}{APP_1} + \frac{MPP_2}{APP_2} + \ldots + \frac{MPP_n}{APP_n}$$

E is defined as the proportional change in output resulting from a unit proportional change in all inputs. Suppose all x_i are increased by the amount
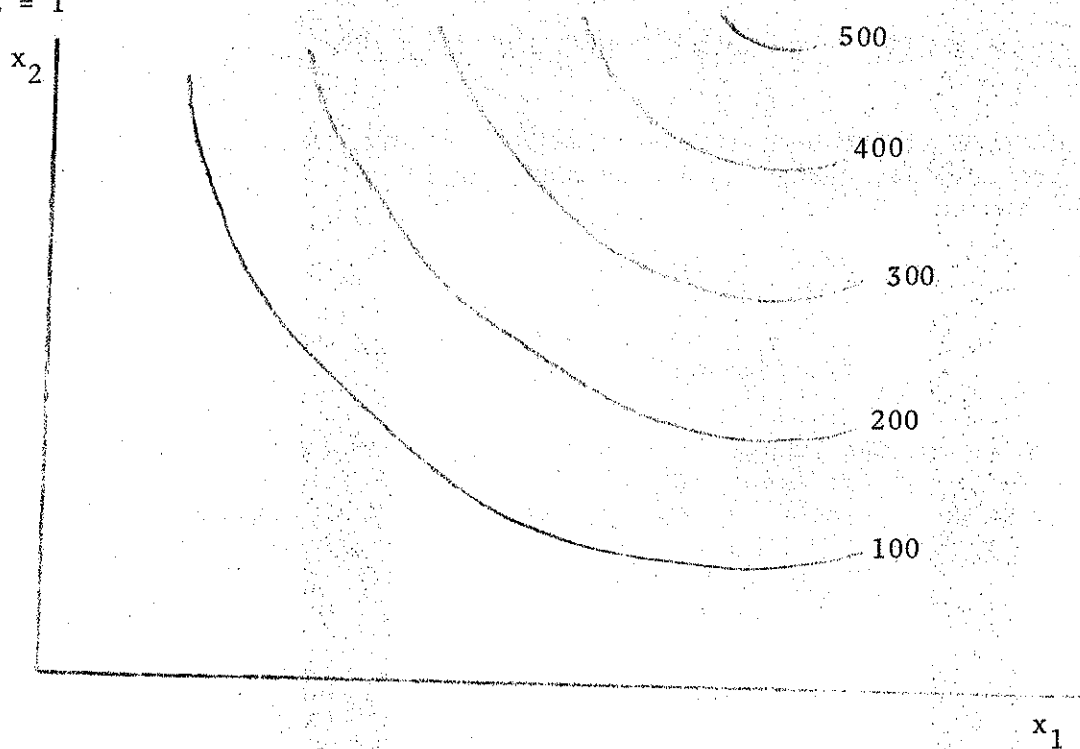
$$\frac{\Delta x_i}{x_i} = k$$

E is the percentage change in output given that the percentage change in all inputs is the same. In other words, if we increase the use of all inputs by 1 percent, by what percent does output change?
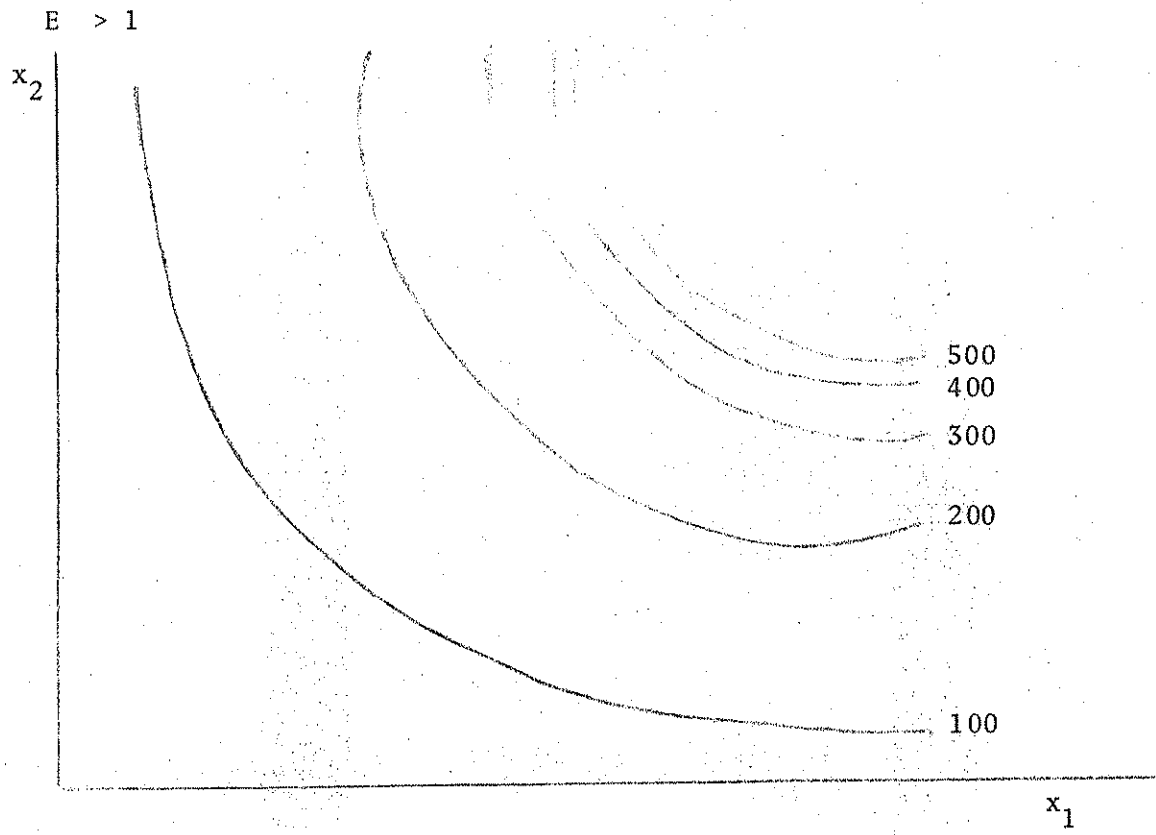
Effect of alternative values of E on isoquants:

$0 < E < 1$
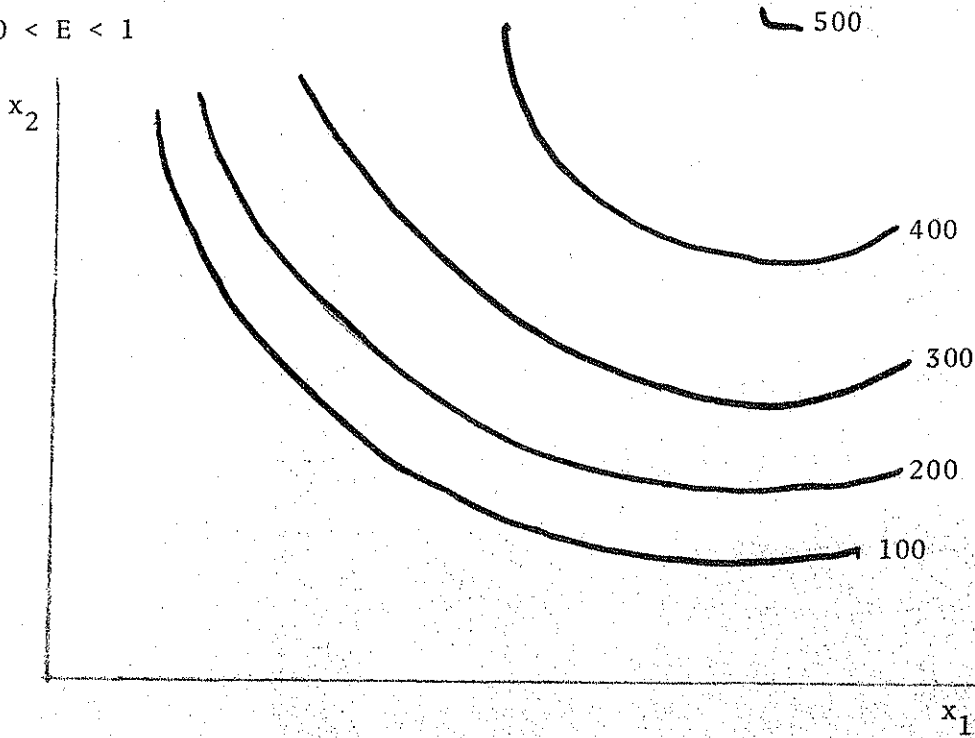
$E = 1$

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Exercise 8

1. Suppose that a production function is given by:

(1) $\quad y = A \, x_1^{\alpha} \, x_2^{1-\alpha}$

Draw the isoquant patterns implied by (1).

2. Given the above production function, what can be deduced with respect to the nature of the 2 underlying single input production functions. Do these single input functions exhibit increasing, constant or decreasing marginal returns to the variable input?

3. Given a production function of the general form:

$$y = Ax_1{}^{\beta_1} x_2{}^{\beta_2}$$

What values would β_1 and β_2 need to assume if the function

(1) Exhibited constant returns to scale (E=1), but diminishing marginal returns to x_1 and x_2.

(2) Exhibited increasing returns to scale in the region
$$1 < E < 2$$
but diminishing marginal returns to x_1 and x_2.

(3) Exhibited diminishing marginal returns to x_1 and x_2 and diminishing returns to scale.

(4) Exhibited constant returns to scale, but diminishing marginal returns to x_1 and x_2.

(5) Exhibited increasing returns to scale in the region E > 2.

(6) Exhibited increasing marginal returns to x_1, diminishing
marginal returns to x_2 and $1 < E < 2$.

4. Given a production function:

$$y = Ax_1^{\alpha} \, x_2^{1-\alpha}$$

show that

$$E = \frac{MPP_1}{APP_1} + \frac{MPP_2}{APP_2} = 1$$

5. Show that for any multiplicative production function such as:

$$y = Ax_1^{\beta_1} \, x_2^{\beta_2} \ldots x_n^{\beta_n}$$

the partial elasticity of production

$$\varepsilon^i = \frac{MPP_i}{APP_i} = \beta_i$$

and that

$$E = \sum_{i=1}^{n} \beta_i$$

6. Under what conditions would the production function:

(1) $y = Ax_1^{\beta_1} \, x_2^{\beta_2} \, \ldots \, x_n^{\beta_n}$

exhibit constant returns to scale?

Under what conditions would (1) exhibit increasing returns to scale?

Under what conditions would (1) exhibit decreasing returns to scale?

Derivation of Kuhn Tucker Conditions

Brief Motivation

Suppose we are producing a product y using 2 inputs x_1 and x_2
the production function is:

$$(1) \quad y = f(x_1, x_2)$$

we are constrained in that we do not wish to spend more than b

dollars on x_1 and x_2

Hence, our constraint is:

$$(2) \quad g(x_1, x_2) \leq b$$

Specifically, we want to be able to spend less than b dollars on x_1

and x_2, if that is optimal given the other resources on our farm.

Add a positive "slack" variable to (2) so that

$$(3) \quad g(x_1, x_2) + u = b$$

therefore:

$$(4) \quad g(x_1, x_2) + u - b = 0$$

Now maximize (1) subject to the constraint

$$(5) \quad L = f(x_1, x_2) - \lambda \, [g(x_1, x_2) + u - b)]$$

$$(6) \quad \frac{\partial L}{\partial x_1} = \frac{\partial f}{\partial x_1} - \lambda \frac{\partial g}{\partial x_1} = 0$$

$$(7) \quad \frac{\partial L}{\partial x_2} = \frac{\partial f}{\partial x_2} - \lambda \frac{\partial g}{\partial x_2} = 0$$

$$(8) \quad \frac{\partial g}{\partial \lambda} = g(x_1, x_2) - u - b = 0$$

$$(9) \quad \frac{\partial g}{\partial u} = -\lambda = 0$$

Multiply (9) times u

$$(10) \quad - \lambda u = 0$$

Since, by (8)

$$(11) \quad u = b - g(x_1, x_2)$$

Substituting into 10

$$(12) \quad - \lambda (b - g(x_1, x_2))$$

Suppose $(b - g(x_1, x_2)) > 0$

hence, not all of b is being used. Then the shadow price or implicit worth of an extra unit of b (λ) is zero, i.e., extra labor is worth nothing if you already have more than enough for the other resources you have.

Suppose, however, that

$$(b - g(x_1, x_2)) = 0$$

then, condition (12) is satisfied if λ is not equal to zero, and is the "implicit worth" of an extra unit of b, the increase in the amount of y that can be produced with an additional unit of b.

Some alternative forms of $f(x_1, x_2)$ that are used in research in agricultural economics:

(1) Multiplicative:

$$Y = x_1 \cdot x_2$$

(What is E for the above function?)

(2) <u>Cobb Douglas</u>[1]:

A specialized case of a wide variety of multiplicative production functions:

$$(1) \quad Y = A \; x_1^{\alpha} \; x_2^{1-\alpha}$$

where:

$Y =$ output

$A =$ a multiplicative parameter incorporating, amongst other things, technology

The function is linear in the logs, that is:

$$\log Y = \log A + \alpha \log x_1 + (1-\alpha) \log x_2$$

This is convenient - allows for the empirical estimation using ordinary (classical) least squares procedures since the parameters A and α are constants, not functions of x_1 or x_2.

Marginal products of the Cobb Douglas:

$$\frac{\partial y}{\partial x_1} = \alpha \; A \; x_1^{\alpha-1} \; x_2^{1-\alpha}$$

$$\frac{\partial y}{\partial x_2} = (1-\alpha) \; A \; x_1^{\alpha} \; x_2^{1-\alpha-1}$$

$$RTS = \frac{dx_1}{dx_2} = \frac{\alpha \; A \; x_1^{\alpha-1} \; x_2^{1-\alpha}}{(1-\alpha) \; A \; x_1^{\alpha}1 \; x_2^{1-\alpha-1}}$$

or

$$RTS = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$$

---

[1] H and Q restrict their definitions of the Cobb Douglas to include only the 2 input function where the production elasticities sum to 1. The general multiplicative function is sometimes referred to as a Cobb Douglas type, of which (1) is an example.

Further, since

$$\frac{\partial y}{\partial x_1} = \alpha A x_1^{\alpha-1} x_2^{1-\alpha}$$

and

$$y = A x_1^{\alpha} x_2^{1-\alpha}$$

$$\frac{\partial y}{\partial x_1} = \alpha \frac{y}{x_1} = MPP_{x_1}$$

and

$$APP_{x_1} = \frac{A x_1^{\alpha} x_2^{1-\alpha}}{x_1} = \frac{y}{x_1}$$

hence $APP \neq MPP$ for any value of x_1 if $\alpha \neq 1$

Since $\alpha < 1$, $MPP < APP$

Exercise:

At what value of x_1 would you be in stage III for x_1?

Isoquants for the Cobb Douglas:

Since:

$$(1) \quad Y = A x_1^{\alpha} x_2^{1-\alpha}$$

$$(2) \quad x_1^{\alpha} = \frac{y}{A x_2^{1-\alpha}}$$

(3) $\quad x_1 = \left(\dfrac{y}{A\, x_2^{\,1-\alpha}} \right)^{1/\alpha}$

Similarly

(4) $\quad x_2 = \left(\dfrac{y}{A\, x_1^{\,\alpha}} \right)^{\frac{1}{1-\alpha}}$

Since, along an Isoquant, y is constant, and A is a constant
(3) and (4) define families of   hyperbolas, the
distance from the origin being determined by the value of y.

Diagramatically:
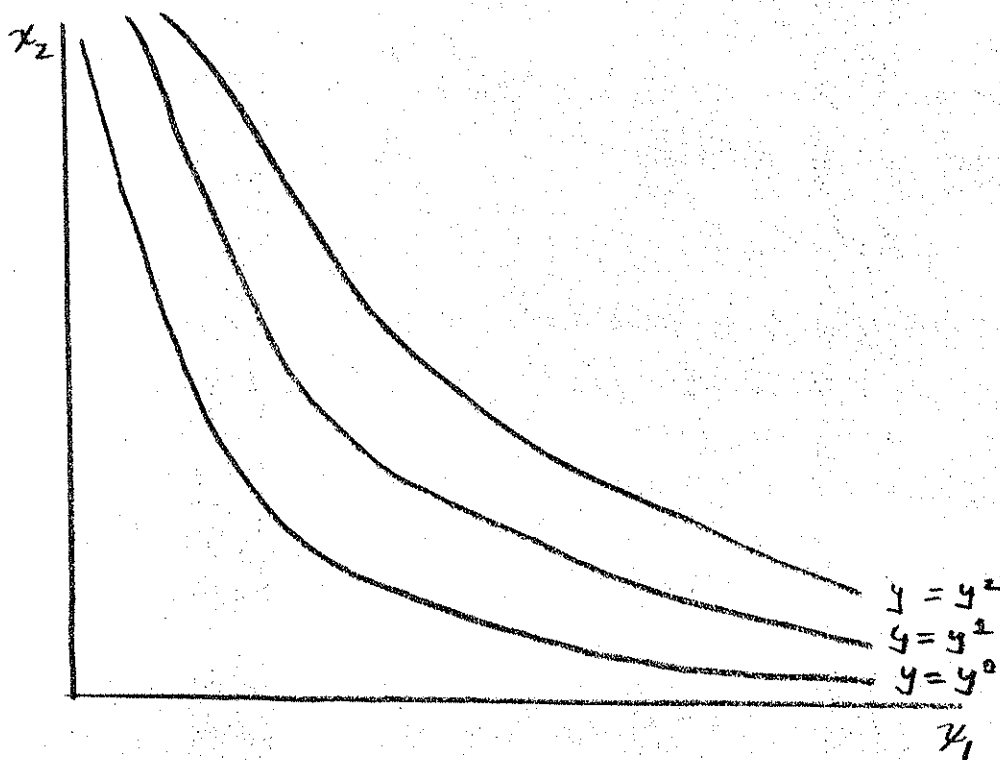
Note that for the Cobb Douglas, Isoquants never curve up. Hence,
there is no stage III.

In other words, we can choose values of x_1 or x_2 as large as we
please, and y will always show an increase as a result.

(3) <u>C.E.S. (Constant elasticity of substitution)</u>

$$Y = A[\alpha \, x_1^{-\rho} + (1-\alpha) \, x_2^{-\rho}]^{-\frac{1}{\rho}}$$

A complicated looking animal

Properties of the C.E.S.:

(a) Homogeneous of degree 1

See H & Q, pg. 86.

(b) Has a constant elasticity of substitution equal to

$$\sigma = \frac{1}{1 + \rho}$$

See characteristics of isoquants

when σ assumes alternative values (H & Q, pp. 87-88).

(c) The Cobb Douglas is a special case of the C.E.S. when $\sigma = 1$. Requires a fancy proof using L'Hôspital's Rule (pp. 87-88 of Henderson and Quandt).

(4) <u>The Variable Elasticity of Substitution Production Function</u>
<center>(VES)</center>

Form:

$$Y = A[\alpha \, x_1^{-\rho} + (1-\alpha) \, \eta \left(\frac{x_1}{x_2}\right)^{-c(1+\rho)} x_2^{-\rho}]^{-\frac{1}{\rho}}$$

if $c = 0$, the VES reduces to the CES.

Allows for variable, not fixed elasticities of substitution.

σ is not fixed, but rather a function of x_1 and x_2. Assuming $c \neq 0$.

Summary

Keep in mind:

	Elasticity of Substitution
Cobb Douglas	1
Cobb Douglas type	1
$\sum_{i=1}^{n} \alpha_i \neq 1$	
CES	$\sigma = \dfrac{1}{1+\rho}$
	A constant that may not be = to 1
VES	σ = not a constant, but a function of x_1 and x_2

(5) Cobb Douglas Production Function with variable returns to scale.[2]

$$Y = AX_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3}$$

where $\alpha_1 = \alpha_1 (z)$

$\alpha_2 = \alpha_2 (z)$

$\alpha_3 = \alpha_3 (z)$

where z is a variable hypothesized to influence the α_i.

Proposed that z could be represented by different managerial ability, different types of capital, or different qualities of labor.

---

[2] See: Edwin F. Ulveling and Lehman B. Fletcher, "A Cobb Douglas Production Function with Variable Returns to Scale." Am. J. Agr. Econ.:52, 322-326, May, 1970.

Estimate by log transformation

$$y = A + \alpha_1(z) \quad x_1 + \alpha_2(z) \quad x_2$$

$$+ \quad \alpha_3(z) \quad x_3$$

(6) <u>Transcendental Production Function:</u>

$$y = c \, x^{\alpha_1} e^{\alpha_2 x}$$

$$\frac{dy}{dx} = y(\frac{\alpha_1}{x} + \alpha_2)$$

Can you solve for dy/dx. Maximum output:

$$y(\frac{\alpha_1}{x} + \alpha_2) = 0$$

Hence:

$$x = - \frac{\alpha_1}{\alpha_2}$$

Properties of: $\quad y = c\, x^{\alpha_1}\, e^{\alpha_2 x}$

Value of α_1	Value of α_2	What happens to y
$0 < \alpha_1 \leq 1$	$\alpha_2 < 0$	increases at decreasing rate until $x = -\dfrac{\alpha_1}{\alpha_2}$, then decreases
$\alpha_1 > 1$	$\alpha_2 < 0$	increases at increasing rate until $x = \dfrac{-\alpha_1 + \sqrt{\alpha_1}}{\alpha_2}$, increases at decreasing rate until $x = -\dfrac{\alpha_1}{\alpha_2}$, then decreases
$0 < \alpha_1 < 1$	$\alpha_2 = 0$	increasing at decreasing rate
$\alpha_1 = 1$	$\alpha_2 = 0$	increases at constant rate
$\alpha_1 > 1$	$\alpha_2 = 0$	increases at increasing rate
$0 < \alpha_1 < 1$	$\alpha_2 > 0$	increases at decreasing rate until $x = \dfrac{-\alpha_1 + \sqrt{\alpha_1}}{\alpha_2}$, then increases at increasing rate
$\alpha_1 \geq 1$	$\alpha_2 > 0$	increases at increasing rate

When $\alpha_2 < 0$, the function appears to be equivalent to the 3 stage "textbook production function." When $\alpha_2 = 0$ the function is a <u>Cobb Douglas type</u> with a constant production elasticity equal to α_1.

See: A.N. Halter, H. O. Carter, and J.G. Hocking. "A Note on the Transcendental Production Function." <u>J. Farm Econ.</u>39: 966-974, Nov., 1957.

Some tabulated values for the Transcendental:

$$y = cx^{\alpha_1} e^{\alpha_2 x}$$

Suppose:

$$c = 1$$

$$\alpha_1 = 4$$

$$\alpha_2 = -2$$

then

$$y = x^4 e^{-2x}$$

$$\frac{dy}{dx} = 4x^3 e^{-2x} - 2e^{-2x} x^4 = 0$$

$$= e^{-2x}(4x^3 - 2x^4) = 0$$

Tabulated Values

x	e^{-2x}	x^4	y	Δy
0	1.0	0	0.000	
				.002
.25	.606	.004	.002	
				.021
.50	.368	.063	.023	
				.047
.75	.223	.316	.070	
				.065
1.00	.135	1.0	.135	
				.065
1.25	.082	2.44	.200	
				.052
1.50	.050	5.06	.252	
				.031
1.75	.030	9.38	.283	
				.010
2.00	.018	16.0	.293	
				-.009
2.25	.011	25.63	.284	
				-.021
2.50	.007	39.06	.263	
				-.030
2.75	.004	57.19	.233	
				-.033
3.00	.002	81.0	.200	

Note that

$$e^{-2x}(4x^3 - 2x^4) = 0 \qquad \text{(the function assumes its maximum}$$
$$\text{when } x = 2 \text{ or}$$

$$x = \frac{\alpha_1}{\alpha_2} = 2 \text{)}$$

Note also that all three stages of production are represented. The function increases at an increasing rate until

$$x = \frac{-\alpha_1 + \sqrt{\alpha_1}}{\alpha_2} = \frac{-4 + \sqrt{4}}{-2} = 1$$

increases at a decreasing rate until,

$$x = \frac{-\alpha_1}{\alpha_2} = 2$$

then decreases.

(7) <u>Generalized Power Production Function</u>[1]

$$Y = Ax_1^{f^1(x_1, x_2)} x_2^{f^2(x_1, x_2)} e^{g(x_1, x_2)}$$

or, more generally

$$Y = A \prod_{k=1}^{k} X_k^{f^k(x_1, \ldots x_k)} \cdot e^{g(x_1 \ldots x_k)}$$

Special Cases.

(a) $g(x_1, x_2) = 0$ and $f^1 = \alpha_1$, $f^2 = \alpha_2$,

                        results in a Cobb Douglas type.

(b) $g(x_1, x_2) = 0$ and $f^1 \neq \alpha_1$, $f^2 \neq \alpha_2$

                    but $f^1 = f^1(x_1, x_2)$

                    and $f^2 = f^2(x_1, x_2)$

                    and f^1, f^2, homogeneous of degree zero
                            in x results in Cobb Douglas
                            with variable returns to scale.

(c) $g(x_1, x_2) = \gamma_1 x_1 + \gamma_2 x_2$

                    and $f^1 = \alpha_1$, $f^2 = \alpha_2$

                        results in the Transcendental.

---

[1] See Alain De Janvry, "The Generalized Power Production Function," <u>Am. J. Agr. Econ.</u> 54: 234-237, May, 1972. Also, for an application see: Alain De Janvry, "Optimal Levels of Corn Fertilizer Under Risk," <u>Am. J. Agr. Econ.</u> 1-10, February, 1972.

The Generalized Power Production Function (GPPF) can describe all three stages of production if $g(x_1, x_2) \neq 0$.

De Janvry works out the Marginal Productivities for the GPPF in the cited article.

VMP vs. MVP

Earlier in the course, we defined VMP as MPP · P.

Sometimes,

$$P \neq a \text{ constant}$$

Rather $\qquad P = \psi(y)$

and $\qquad \psi' < 0$

hence, a demand function of a negative slope is indicated.

Lets start with a definition of TVP in this case:

$$TVP = TPP \cdot P$$

Hence:

$$TVP = TPP \cdot \psi(y)$$

but, there is also some underlying production function

$$y = \phi(x)$$

Hence:

$$TVP = y \cdot \psi(\phi(x))$$

A rather complicated looking bird to say the least!

Define

MVP is the change in TVP with respect to a change in input use

Hence: $\qquad MVP = \dfrac{dTVP}{dx}$

Given that P = a constant

$$TVP = py$$

and $\quad MVP = \dfrac{pdy}{dx} = \dfrac{dTVP}{dx}$

Now, however, since

$$TVP = TPP \cdot \psi \ (\phi(x))$$

$$\frac{dTVP}{dx} = y \cdot \frac{d\psi}{d\phi} \frac{d\phi}{dx} + P \frac{dy}{dx}$$

Hence:

$$MVP = P \underbrace{\frac{dy}{dx}}_{} + \underbrace{Y \frac{d\psi}{d\phi} \frac{d\phi}{dx}}_{}$$

Old	A real
definition	animal
of VMP	

What is:

$$Y \ \frac{d\psi}{d\phi} \ \frac{d\phi}{dx}$$

We all know Y is TPP

$$\frac{d\phi}{dx} \equiv \frac{dy}{dx} \ \text{since } y = \phi(x)$$

$\dfrac{d\psi}{d\phi}$ is the change in price associated with a change in the output level.

So

$$MVP = P \cdot MPP + Y \frac{d\psi}{d\phi} \ MPP$$

$$MVP = MPP \ [P + Y \frac{d\psi}{d\phi}]$$

$$MVP = MPP \cdot P \quad [1 + \frac{y}{p}\frac{d\psi}{d\phi}]$$

but $\frac{d\psi}{d\phi}$ is really $\frac{dp}{dy}$ so $MVP = MPP \cdot P [1 + \frac{y}{p}\frac{dp}{dy}]$

$$MVP = MPP \cdot P [1 + \frac{1}{E_d}]$$

Hence:

$$MVP = VMP [1 + 1/E_d]$$

Exercise:

Given the above formula, why is it that for a demand curve for the output that is a horizontal, straight line, MVP = VMP?

MVP is usually ≤ 0 (i.e., a downsloping demand curve)

A Picture:
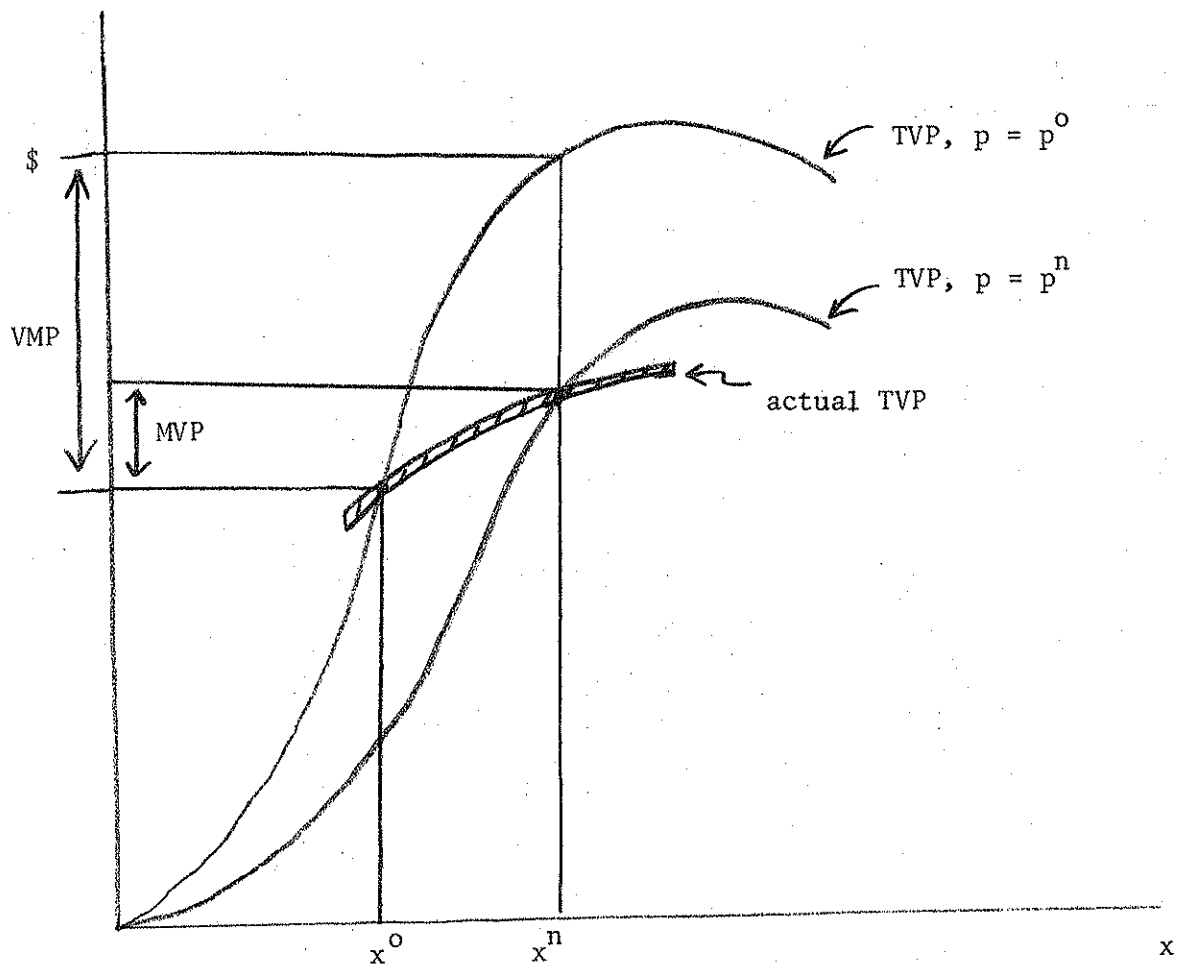
Marginal Resource Cost

We can do the same thing for MRC.

Suppose TRC = vx

if v = constant

$$MRC = \frac{dTRC}{dx} = v$$

However, suppose

$$v = \theta(x)$$

In particular, suppose

$$\frac{dv}{dx} = \frac{d\theta}{dx} < 0$$

or

$$\frac{dv}{dx} = \frac{d\theta}{dx} > 0$$

What real world situations would result in these conditions?

Then   TRC = $\theta(x)$ x

$$\frac{dTRC}{dx} = \theta \frac{dx}{dx} + x \frac{d\theta}{dx}$$

Hence:

$$MRC = v + x \frac{dv}{dx}$$

Why?

MRC = price of input +

amount of input x •

change in input price with
respect to a change in input use.

$$MRC = v(1 + \frac{1}{E_x})$$

MRC > v if $E_x > 0$

MRC < v if $E_x < 0$

MRC = v if $E_x = \infty$

What is the slope of the MRC curve?

We need to find $\dfrac{dMRC}{dx}$.

We know that if

$$\frac{dMRC}{dx} < 0:$$

We have
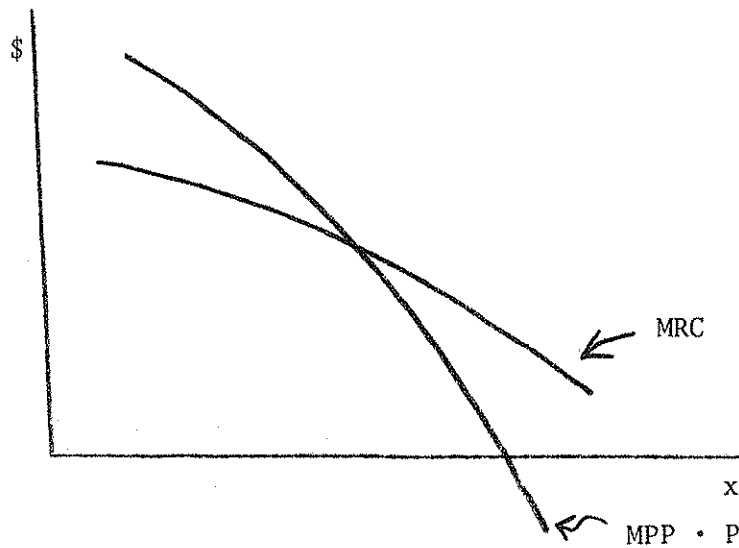
$$\frac{dMRC}{dx} = 0:$$

$$\frac{dMRC}{dx} > 0:$$
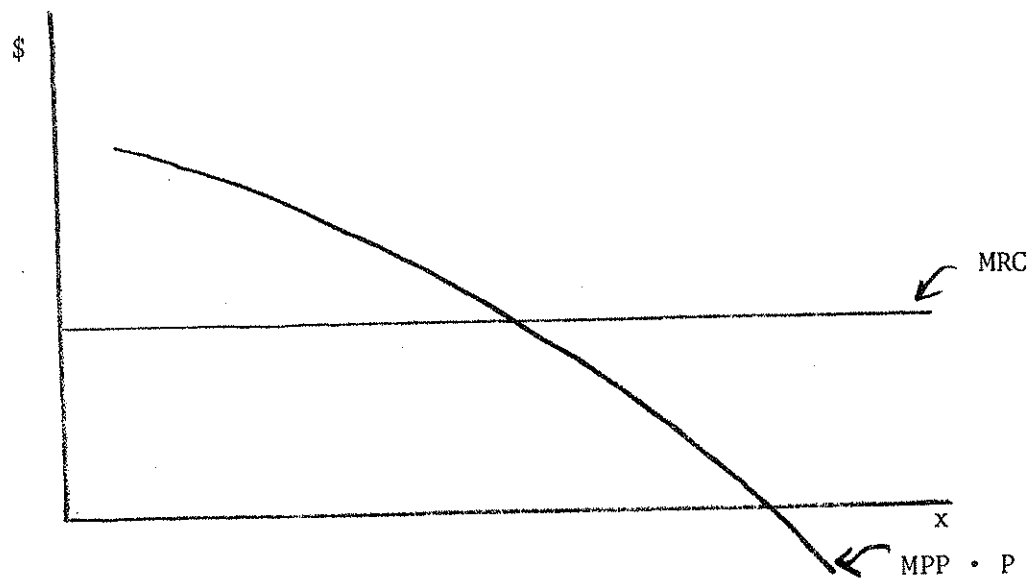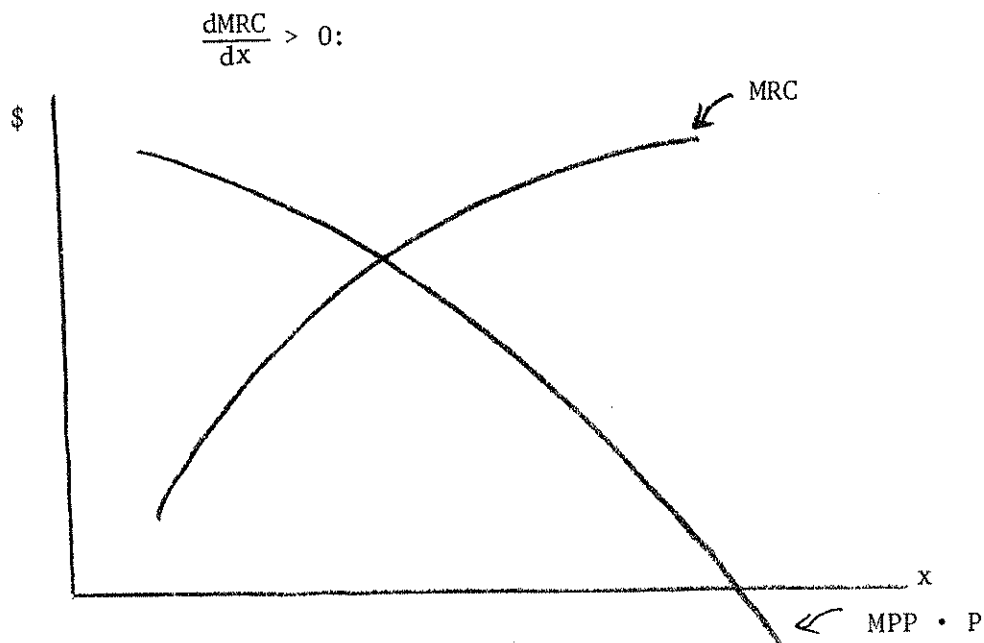
Technical Factor Interdependence

Factors are interdependent if the marginal productivity function for one factor shifts if the quantity of the second factor varies.

A picture:

(An increase in x_2 increases the MPP of x_1)
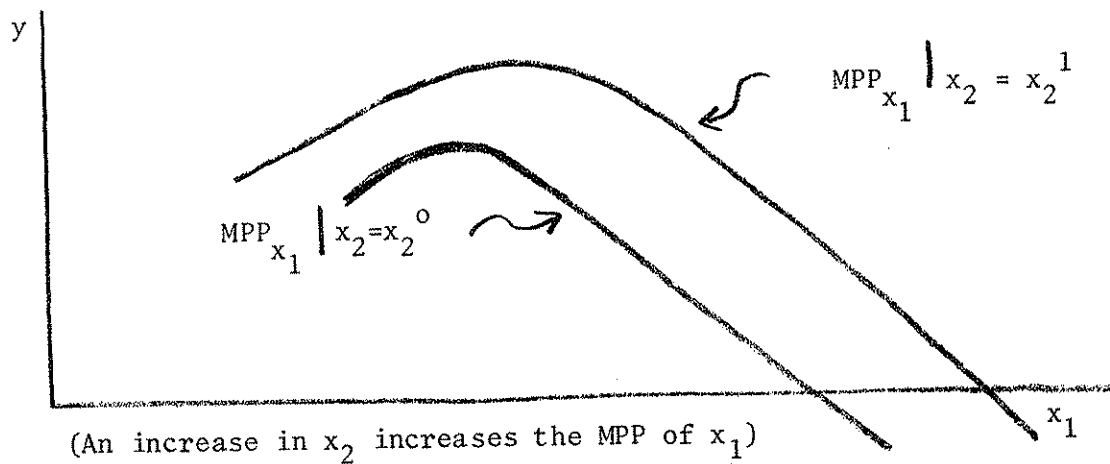
The above picture illustrates technical complementarity assuming

$$x_2^1 > x_2^o$$

$$\frac{\partial \left(\frac{\partial y}{\partial x_1} \right)}{\partial x_2} > 0$$

Excerise:

For the Cobb Douglas type function, are the inputs x_1 and x_2 -- competitive, independent or complimentory given:

$$y = a \, x_1^{\beta_1} \, x_2^{\beta_2}$$

How about:

$$y = a \, x_1 + bx_2$$

How about:

$$y = x_1^2 \, e^{-x_1} \, x_2^3 \, e^{-x_2}$$

Technical independence
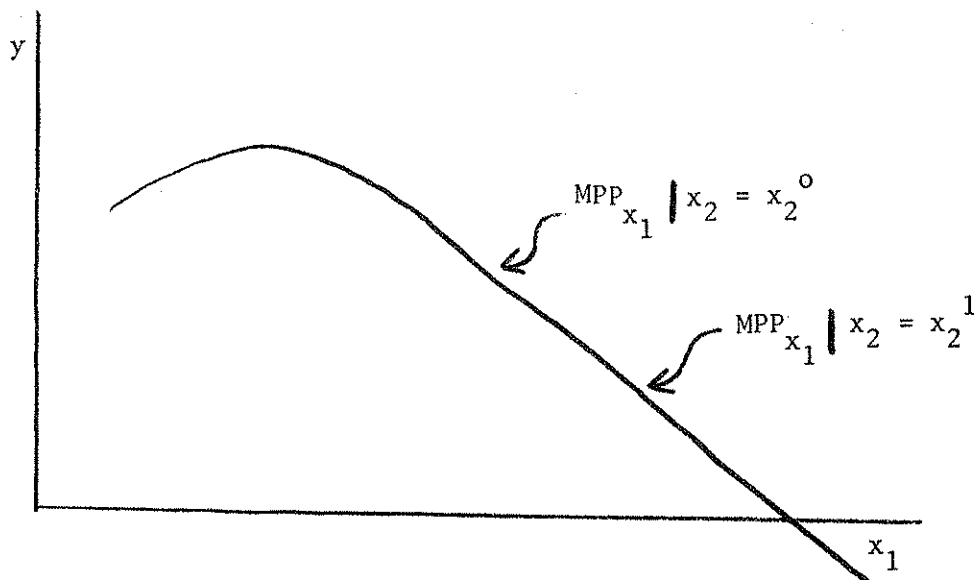
$$\frac{\partial \left(\frac{\partial y}{\partial x_1} \right)}{\partial x_2} = 0$$

(An increase in x_2 does not change the MPP of x_1)

Technical competitiveness
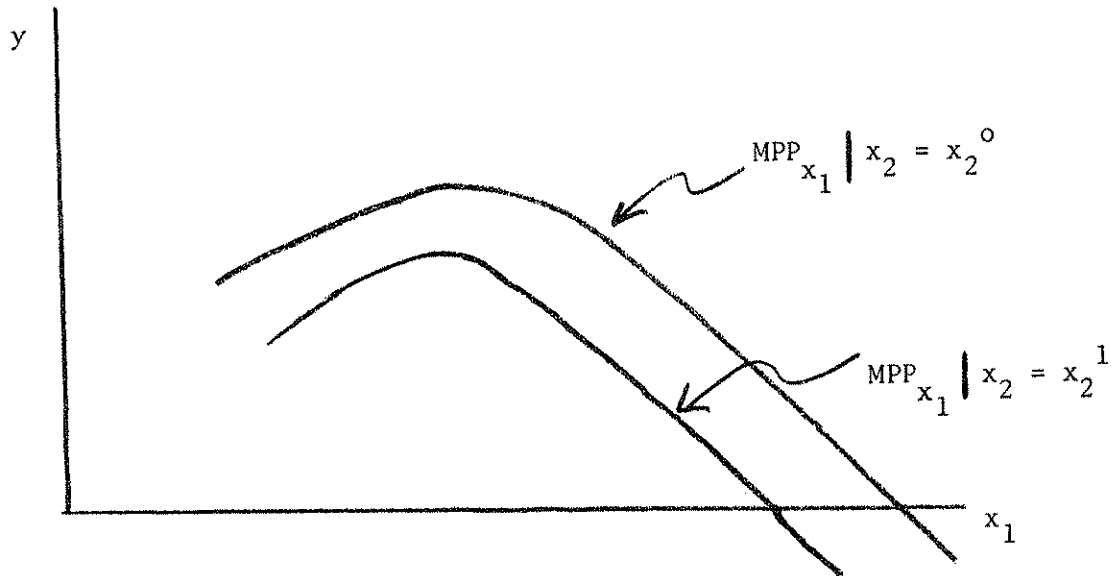
The above picture illustrates technical competitiveness assuming

$$x_2^1 > x_2^o \qquad \frac{\partial\left(\frac{\partial y}{\partial x_1}\right)}{\partial x_2} < 0 \quad \text{an increase in } x_2 \text{ decreases the MPP of } x_1.$$

Homogeneous functions and Eulers theorem

Definition: A function is defined as being homogeneous of degree
n in the variables if:

$$(1) \quad f(tx_1, \ldots tx_m) = t^n f(x_1, \ldots x_m)$$

Example:

$$(2) \quad Y = A \, x_1^{\alpha} \, x_2^{1-\alpha}$$

What is the degree of homogeneity?

Multiply x_1 and x_2 by t

$$(3) \quad A \, (tx_1)^{\alpha} \, (tx_2)^{1-\alpha}$$

$$(4) \quad A \, t^{\alpha} \, x_1^{\alpha} \, t^{1-\alpha} \, x_2^{1-\alpha}$$

Summing superscripts on t:

(5) $A t^1 x_1^{\alpha} x_2^{1-\alpha}$

∴ (6) $t^n Y = t^1 A x_1^{\alpha} x_2^{1-\alpha}$

where $n = 1$

Hence: (2) is homogeneous of degree 1 in the inputs x_1 and x_2.

A doubling of x_1 and x_2 will yield twice the output.

Consumer demand functions are often specified to be

homogeneous of degree zero in prices and income that is,

if:

$$Z_1 = \phi(P_1, \ldots P_m, I)$$

then

$$t^o Z_1 = \phi(t P_1, \ldots t P_m, t I)$$

where $n = 0$

A doubling of all prices and income will yield no increase in the demand for Z_1.

Exercise:

What is the degree of homogeneity of the following functions representing the demand for Z_1.

(1) $Z_1 = \dfrac{I^2}{P_1 P_2}$

(2) $Z_1 = \dfrac{I}{P_1 P_2}$

(3) $Z_1 = \dfrac{P_2}{P_1} \cdot I$

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Some functions are not homogeneous.

For Example:

$$Y = x_1 + x_2^2$$

$$(tx_1)^1 + (tx_2)^2$$

Can't factor out t.

Exercise:

Is the function $Y = Ax_1^{\alpha} x_2^{2-\alpha}$ homogeneous?

Euler's Theorem

Euler's theorem states that for a function homogeneous of degree n, the following relation holds.

$$(1) \quad x_1 \frac{\partial f}{\partial x_1} + \ldots + x_m \frac{\partial f}{\partial x_m} = ny$$

Proof:

We begin with the initial relation

$$(2) \quad y = f(x_1, \ldots x_m)$$

Now let

$$(3) \quad x_1 = a_1 t$$

$$x_2 = a_2 t$$

$$x_m = a_m t$$

We know that for homogeneous functions:

$$(4) \quad f(a_1 t, a_2 t, \ldots a_m t) =$$

$$(5) \quad t^n f(a_1, a_2, \ldots a_m)$$

Since

$$(6) \quad y = f(x_1, x_2, \ldots x_m)$$

$$(7) \quad y = f(a_1 t, a_2 t, \ldots a_m t)$$

and

$$(8) \quad t^n f(a_1, a_2, \ldots a_m)$$

the total derivative of (7) with respect to t is

$$(9) \quad \frac{dy}{dt} = \frac{\partial f}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \frac{dx_2}{dt} \ldots + \frac{\partial f}{\partial x_m} \frac{dx_m}{dt}$$

$$(y = f(x), \quad x = g(t) \text{ since } x = at)$$

Since $x_i = a_i t$:

(10) $\dfrac{dx_i}{dt} = a$

and:

(11) $\dfrac{dy}{dt} = \dfrac{\partial f}{\partial x_1} a_1 + \dfrac{\partial f}{\partial x_2} a_2 \ldots + \dfrac{\partial f}{\partial x_m} am$

But also, differentiating (8)

(12) $\dfrac{dy}{dt} = nt^{n-1} f(a_1, a_2, \ldots a_m)$

or Setting (11) and (12) equal and multiplying by t yields:

(13) $\dfrac{\partial f}{\partial x_1} a_1 t + \dfrac{\partial f}{\partial x_2} a_2 t + \ldots + \dfrac{\partial f}{\partial x_m} a_m t =$

$nt^n f(a_1, a_2 \ldots a_m)$

or

(14) $\dfrac{\partial f}{\partial x_1} x_1 + \dfrac{\partial f}{\partial x_2} x_2 + \ldots + \dfrac{\partial f}{\partial x_m} x_m =$

$nt^n f(a_1, a_2, \ldots a_m)$

but

$nt^n f(a_1, a_2, \ldots a_m) =$

$n f(ta_1, ta_2, \ldots ta_m) =$

$n f(x_1, x_2, \ldots x_m) = ny$

Hence:

$$\dfrac{\partial f}{\partial x_1} x_1 + \dfrac{\partial f}{\partial x_2} x_2 \ldots + \dfrac{\partial f}{\partial x_m} x_m = ny$$

Q.E.D.

Note that for a function homogeneous of degree 1

$$\frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \ldots + \frac{\partial f}{\partial x_m} x_m = y$$

$$MPP_{x_1} \cdot x_1 + MPP_{x_2} x_2 + \ldots + MPP_{x_m} x_m = y$$

Product Exhaustion Theorem

MPP_{x_i} could be the relative price of the input x_i.

Fundamental Truth - For a production function homogeneous of degree 1, if each resource is paid its marginal product, total product will just be exhausted.

Exercise:

Will the product be exhausted if the production function is homogeneous of a degree greater than one? What about less than one?

It follows from the product exhaustion theorem that long run profit will equal zero.

(1) $\quad \frac{\partial f}{\partial x_1} x_1 + \frac{\partial f}{\partial x_2} x_2 + \ldots + \frac{\partial f}{\partial x_m} x_m = y$

(2) $\quad p \frac{\partial f}{\partial x_1} x_1 + p \frac{\partial f}{\partial x_2} x_2 + \ldots + p \frac{\partial f}{\partial x_m} x_m = yp$

(3) but $\quad p \frac{\partial f}{\partial x_i} = v_i$ from f.o.c.

hence:

(4) $\quad v_1 x_1 + v_2 x_2 \ldots + v_m x_m = yp$

Input Demand Functions express demand for inputs as a function of input and output prices.

$$y = ax_1^{\alpha}$$

$$\Pi = py - v_1 x_1$$

$$\Pi = pax_1^{\alpha} - v_1 x_1$$

$$\frac{d\Pi}{dx_1} = \alpha \, pax_1^{\alpha-1} - v_1 = 0$$

(Π max condition)

$$\alpha \, pax_1^{\alpha-1} = v$$

$$x_1^{\alpha-1} = \frac{v_1}{\alpha pa}$$

$$x_1 = \left(\frac{v_1}{\alpha pa}\right)^{\frac{1}{\alpha-1}}$$

I. Can we ascertain what the impact of (1) an increase in the price of the demand for x_1 input v. Clearly, if $\alpha < 1$, $\alpha-1 < 0$.
Hence:

$$x_1 = v_1^{\frac{1}{\alpha-1}} \, p^{\frac{-1}{\alpha-1}} \, a^{\frac{-1}{\alpha-1}} \, \alpha^{\frac{-1}{\alpha-1}}$$

Hence:

$$\frac{dx_1}{dv_1} \; < \; 0$$

Since v_1, p, α, a, all > 0 and

$$\frac{1}{\alpha-1} \; < \; 0$$

No real surprise as the price of an input goes up, less is demanded.

II. What is the impact of an increase in product price on the demand for an input?

$$\frac{dx_1}{dp} \; > \; 0 \quad \text{since}$$

v_1, p, α, a, all > 0

and $\quad -\dfrac{1}{\alpha-1} \; < \; 0$

It has been widely disseminated that the demand for an input to the production process is merely the demand for the final good less a processing margin.[1]

Hence:
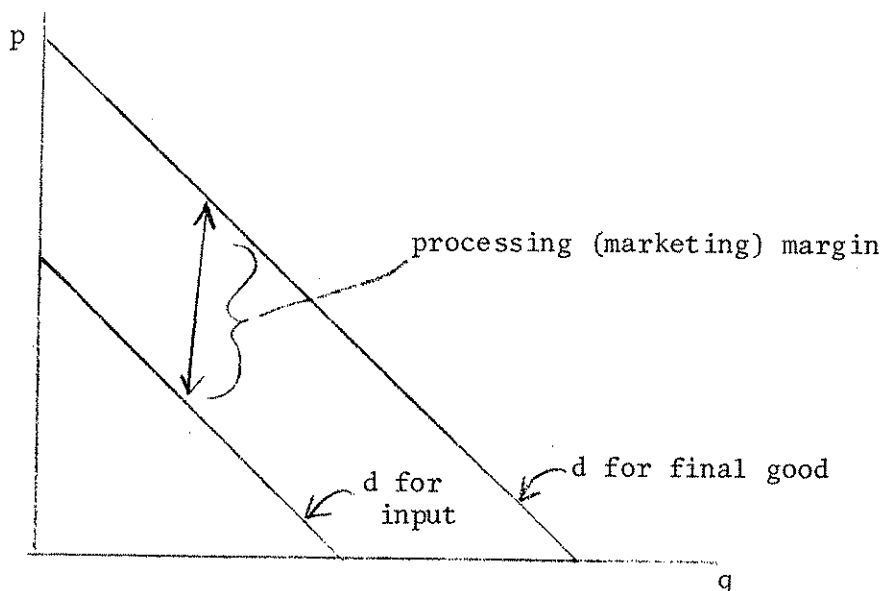
Figure 1.

This implies that the demand for the input is inversely related to the product price. Hence, $\dfrac{dx_1}{dp} < 0.$ Under the assumption of a multiplicative Cobb Douglas type production function. Clearly this is not the case.

$$\frac{dx_1}{dp} > 0$$

Figure 1 is nonsense!

---

[1] See: R.C. Haidacher, "Some Suggestions for Developing New Models from Existing Models," Am. J. Agr. Econ. 52:5, December, 1970, pp. 814-819, especially 817-818.

The 2 input case:

I. VMP demand

tracing out the demand function for the input x_1 along the VMP curve.

Production function:

$$y = ax_1^{\alpha} x^{\alpha}$$

$$2\ \alpha < 1$$
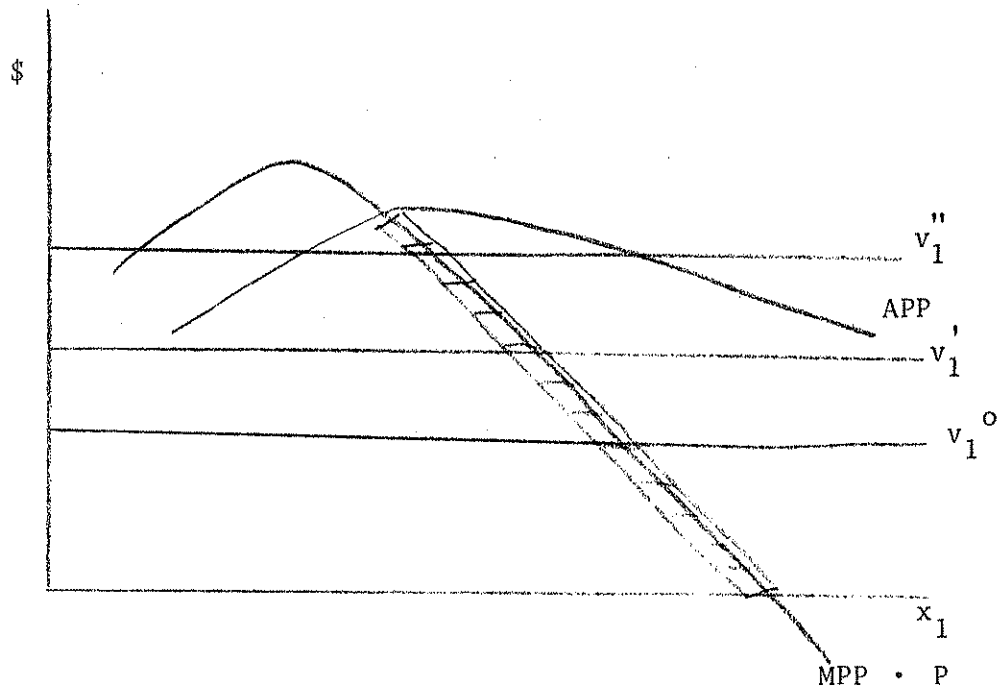
Could use α_1 and α_2 $\alpha_1 \neq \alpha_2$ but not necessary to argument.

Math is simplified.

Profit function:

$$\Pi = pax_1^{\alpha} x_2^{\alpha} - v_1 x_1 - v_2 x_2$$

$$\frac{\partial \Pi}{\partial x_1} = \alpha\ pax_1^{\alpha-1} x_2^{\alpha} - v_1 = 0$$

Exercise 9:

1. Suppose a production function and cost function is given as:

$$y = ax_1^3$$

$$c = vx_1$$

What can be deduced with respect to the elasticity of demand for the input x_1?

2. Suppose a production function and cost function is given as:

$$y = ax_1$$

$$c = vx_1$$

What can be deduced with respect to the elasticity of demand for the input x_1?

3. Suppose a production function and cost function is given as:

$$y = ax_1^{.6}$$

$$c = vx_1$$

What can be deduced with respect to the elasticity of demand for the input x_1?

$$\frac{\partial \Pi}{\partial x_2} = \alpha \; pax_1^{\alpha} x_2^{\alpha-1} - v_2 = 0$$

$$x_1 = \left(v_1\right)^{\frac{1}{\alpha-1}} \left(\alpha\right)^{\frac{-1}{\alpha-1}} \left(p\right)^{\frac{-1}{\alpha-1}} \left(x_2\right)^{\frac{-\alpha}{\alpha-1}}$$

Hence:

$$x = \phi(p, \; x_2, \; v_1)$$

Wish to find:

$$\frac{\partial x_1}{\partial p} \; , \quad \frac{\partial x_1}{\partial x_2} \; , \quad \frac{\partial x_1}{\partial v_1}$$

(1) $$\frac{\partial x_1}{\partial p} = \left(-\frac{1}{\alpha-1}\right)\left(p\right)^{\frac{-1}{\alpha-1}-1} \left(q\right)$$

where $q = \left(v_2\right)^{\frac{1}{\alpha-1}} \left(\alpha\right)^{\frac{-1}{\alpha-1}} \left(x_2\right)^{\frac{-\alpha}{\alpha-1}}$

Hence:

$$\frac{\partial x_1}{\partial p} \; > \; 0 \; \text{ Since } \; \frac{-1}{\alpha-1} \; > \; 0$$

(2) $$\frac{\partial x_1}{\partial x_2} = -\frac{\alpha}{\alpha-1}\left(x_2\right)^{\frac{-\alpha}{\alpha-1}-1} r$$

where $r = \left(v_2\right)^{\frac{1}{\alpha-1}} \left(\alpha\right)^{\frac{-1}{\alpha-1}} \left(p\right)^{\frac{-1}{\alpha-1}}$

Hence:

$$\frac{\partial x_1}{\partial x_2} > 0 \; \text{ Since } \; \frac{-\alpha}{\alpha-1} \; > \; 0$$

Use of extra x_1 also requires use of more x_2.

(3) $\dfrac{\partial x_1}{\partial v_1} = \left(\dfrac{1}{\alpha-1}\right)\left(v_1\right)^{\frac{1}{\alpha-1}-1}\left(r\right)$

where

$$r = \alpha^{\frac{-1}{\alpha-1}}\left(p\right)^{\frac{-1}{\alpha-1}}\left(x_2\right)^{\frac{-\alpha}{\alpha-1}}$$

Hence:

$\dfrac{\partial x_1}{\partial v_1} < 0$ Since $\dfrac{1}{\alpha-1} < 0$

As the input price goes up, less of it is used.

II. Profit Maximization Demand

Wish to trace out the demand function for x_1, allowing x_2 to vary.

DEMAND for input

MPP · P
$(x_2 = x_2^{o}$

MPP ·P
$(x_2 = x_2'$
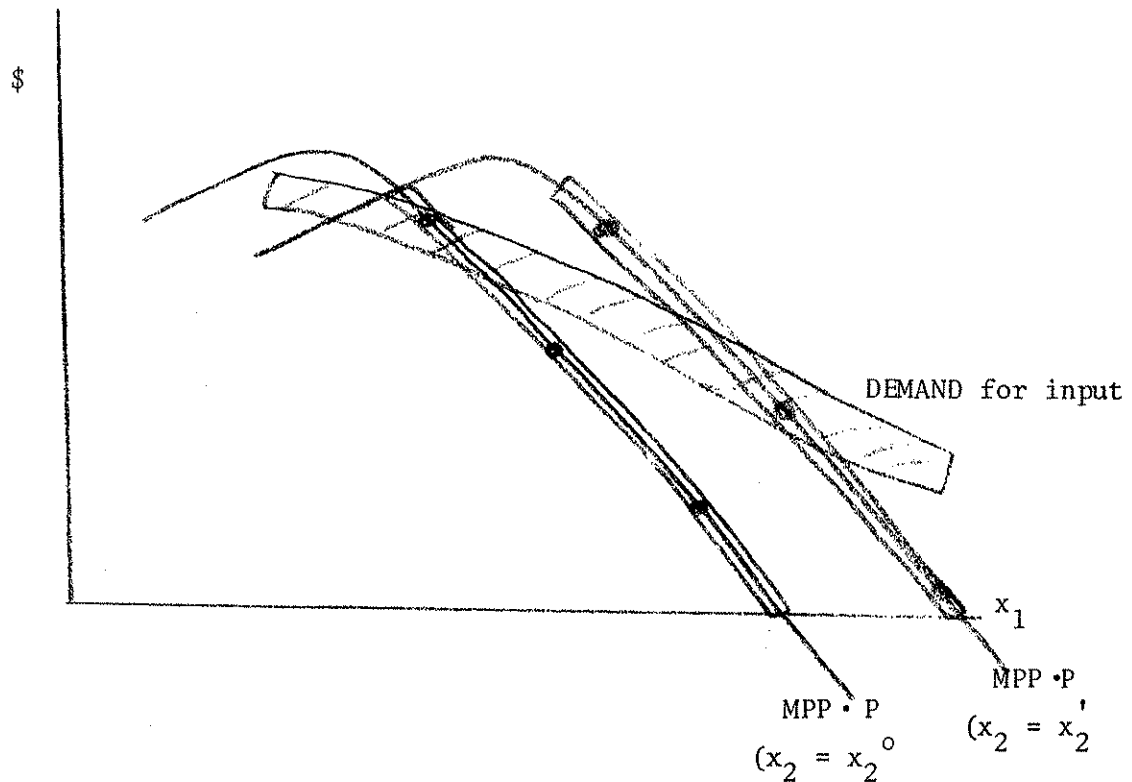
(Are the inputs technically competitive, independent or complementary?)

f.o.c.

(1) $\dfrac{\partial \Pi}{\partial x_1} = \alpha p a x_1^{\alpha-1} x_2^{\alpha} - v_1 = 0$

(2) $\dfrac{\partial \Pi}{\partial x_2} = \alpha p a x_1^{\alpha} x_2^{\alpha-1} - v_2 = 0$

divide (1) by (2)

(3) $x_2 = \dfrac{v_1 x_1}{v_2}$

Substitute (3) into (1)

(4) $\alpha p a x_1^{\alpha-1} v_1^{\alpha} x_1^{\alpha} v_2^{-\alpha} - v_1 = 0$

(5) $\alpha p a x_1^{2\alpha-1} v_2^{-\alpha} = v_1^{1-\alpha}$

(6) $x_1 = (\alpha p a)^{\frac{-1}{2\alpha-1}} v_2^{\frac{+\alpha}{2\alpha-1}} v_1^{\frac{1-\alpha}{2\alpha-1}}$

Wish to determine the sign on

$$\dfrac{\partial x_1}{\partial p}, \quad \dfrac{\partial x_1}{\partial v_2}, \quad \dfrac{\partial x_1}{\partial v_1}$$

$$\dfrac{\partial x_1}{\partial p} > 0 \quad \text{Why?}$$

$$\dfrac{\partial x_1}{\partial v_2} < 0 \quad \text{Why?}$$

$$\dfrac{\partial x_1}{\partial v_1} < 0 \quad \text{Why?}$$

CHAPTER VI

Production of Joint Products

Consider a case where:

$$g(y_1, y_2) = x$$

alternately:

$$\phi(y_1, y_2, x) = 0$$

A production possibilities curve is defined by the equation:

$$x^o = g(y_1, y_2)$$

This is sometimes also referred to as a product transformation curve.

There is a whole family of production possibilities curves (PPC) that can be obtained by specifying alternate values of x.

The Textbook PPC

The textbook PPC is as shown in Figure 1.
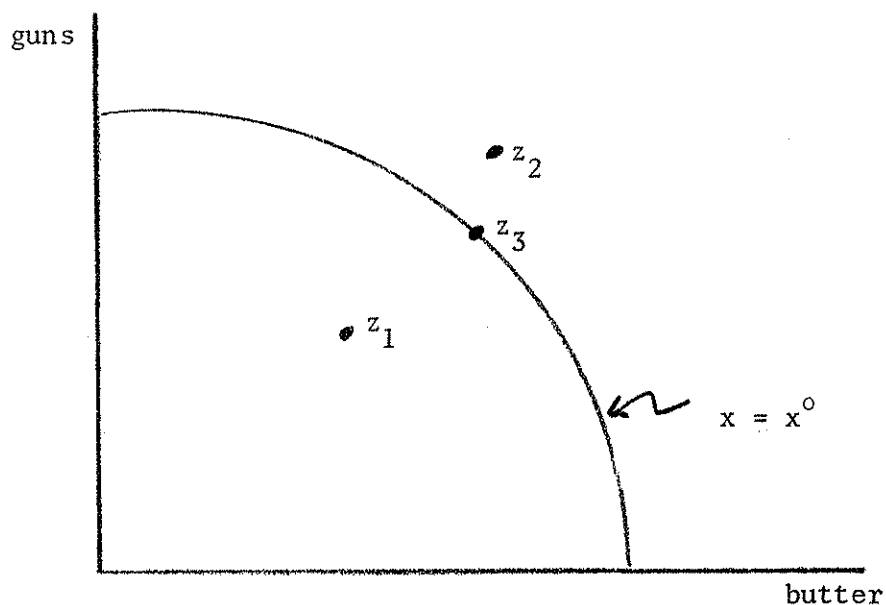
Figure 1.

All of you are familiar with a PPC from your elementary courses. We
are attempting to represent alternative amounts of butter and guns
(y_1 and y_2) that can be produced from a fixed set of resources (x) that
are available to a country.

The PPC encloses a <u>feasible</u> <u>set</u> of possibilities for the country.
Z_1 is feasible, since it lies interior to the PPC. Z_2 is infeasible,
since to produce Z_2 would require more resources than the society has.
Z_3 is feasible, a <u>boundry point "on the frontier."</u>

Note also that Z_2 lies on a new PPC requiring x' amount of resources
where $x' > x^o$. The further a PPC lies away from the origin, the greater
the amount of x that is required.

On a firm level we merely replace butter and guns (y_1 and y_2)
with hogs and beef or any other 2 outputs of interest that compete for
the same input(s).

Proposition:

Whether a PPC is convex to the origin, concave to the origin or
a diagonal straight line depends on whether the 2 production functions
for y_1 and y_2 exhibit increasing, decreasing or constant returns to the
variable input x_1.

<u>Exercise:</u>

1. Graphically derive a production possibilities curve from two production
   functions. Assume that both production functions exhibit <u>diminishing</u>
   <u>marginal returns.</u>

2. Do the same as 1, except assume both production functions exhibit
   <u>constant marginal returns.</u>

3. Do the same as 1, except assume both production functions exhibit
   <u>increasing marginal returns.</u>

The Rate of Product Transformation

The Slope of the PPC is the rate of product transformation (RPT).
In finite increments, it is equal to

$$\frac{\Delta y_2}{\Delta y_1}$$

or, using calculus

$$\frac{dy_2}{dy_1}$$

Suppose

$$x = g(y_1, y_2)$$

$$\Delta x = \frac{\partial g}{\partial y_1} \Delta y_1 + \frac{\partial g}{\partial y_2} \Delta y_2$$

The total differential of x is

$$dx = \frac{\partial g}{\partial y_1} dy_1 + \frac{\partial g}{\partial y_2} dy_2$$

Since, along a PPC, $x^o = a$ constant, $dx^o = 0$

Hence:

$$0 = \frac{\partial g}{\partial y_1} dy_1 + \frac{\partial g}{\partial y_2} dy_2$$

or

$$-\frac{dy_2}{dy_1} = \frac{\partial g / \partial y_1}{\partial g / \partial y_2}$$

The RPT is equal to the ratio of the marginal cost of y_1 in terms
of x to the marginal cost of y_2 in terms of x.

$$\frac{1}{\dfrac{\partial g}{\partial y_1}} \quad = \quad \frac{\partial y_1}{\partial g}$$

$$\frac{1}{\dfrac{\partial g}{\partial y_2}} \quad = \quad \frac{\partial y_2}{\partial g}$$

Hence:

$$RPT = \frac{-dy_2}{dy_1} = \frac{\dfrac{\partial g}{\partial y_1}}{\dfrac{\partial g}{\partial y_2}}$$

$$= \frac{\dfrac{\partial y_2}{\partial g}}{\dfrac{\partial y_1}{\partial g}} \quad = \quad \frac{\dfrac{\partial y_2}{\partial x}}{\dfrac{\partial y_1}{\partial x}}$$

(Why?)

The RPT equals the ratio of the MPP of x in the production of y_2 to the MPP of x in the production of y_1.

Revenue Maximization Subject to an Output Constraint

Deals with the optimum allocation of a fixed set of resources amongst alternative outputs (y_1 and y_2).   TR = $p_1 y_1 + p_2 y_2$ Product transformation curve a locus representing alternative quantities of y_1 and y_2 that can be produced with a fixed amount of x = x^o defined by:

$$x^o = g(y_1, y_2)$$

A Picture:
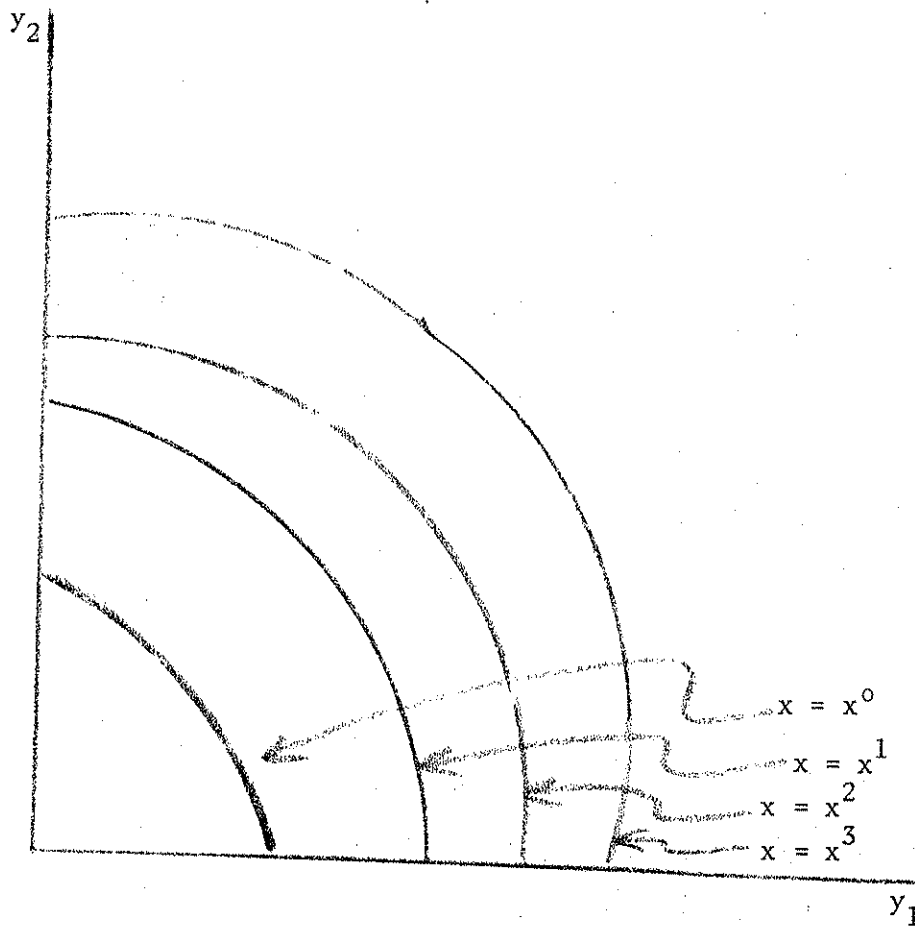
Constraint:

$$x^o = g(y_1, y_2)$$

The Lagrangian:

$$L = p_1 y_1 + p_2 x_2 + \mathcal{U}[x^o - g(y_1, y_2)]$$

where $\mathcal{U}$ is some undetermined lagrangian multiplier

(Increase in total revenue associated with an additional unit

 of the resource x).

The lagrangian has the same function as before.

f.o.c.

$$(1) \quad \frac{\partial L}{\partial y_1} = p_1 - \lambda \frac{\partial g}{\partial y_1} = 0$$

$$(2) \quad \frac{\partial L}{\partial y_2} = p_2 - \lambda \frac{\partial g}{\partial y_2} = 0$$

$$(3) \quad \frac{\partial L}{\partial \lambda} = x^o - g(y_1, y_2)$$

dividing (1) by (2)

$$\frac{p_1}{p_2} = \frac{dy_2}{dy_1}$$

Total revenue max at for any $x = x^o$. Where $\frac{p_1}{p_2}$ = slope of Product

transformation curve.

A picture:
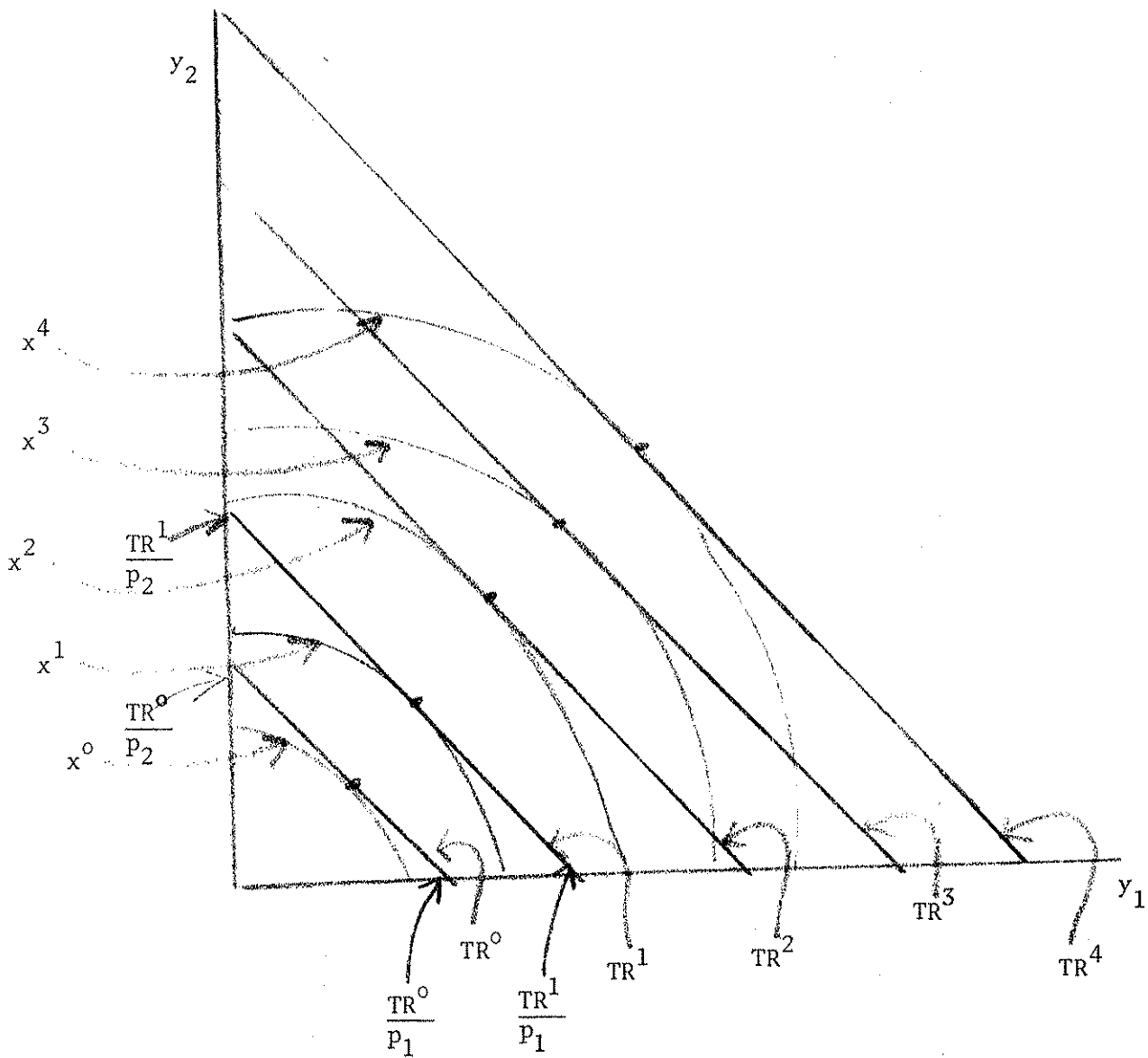
Maximum total revenue for a fixed amount of resource x where slope of TR line is equal to slope of product transformation curve.

Also:

Since

$$p_1 = \frac{\partial g}{\partial y_1}$$

$$p_2 = \frac{\partial g}{\partial y_2}$$

$$\frac{p_1}{\frac{\partial g}{\partial y_1}} = \mathcal{U}$$

$$\frac{p_2}{\frac{\partial g}{\partial y_2}} = \mathcal{U}$$

Since:

$$x = g(y_1, \dot{y}_2)$$

$$\frac{\partial y_1}{\partial x} \cdot p_1 = \mathcal{U}$$

$$\frac{\partial y_2}{\partial x} \cdot p_2 = \mathcal{U}$$

s.o.c.

$$\begin{vmatrix} -\mathcal{U}g_{11} & -\mathcal{U}g_{12} & -g_1 \\ -\mathcal{U}g_{21} & -\mathcal{U}g_{22} & -g_2 \\ -g_1 & -g_2 & 0 \end{vmatrix} > 0$$

Profit Maximization:

$$\Pi = TR - TC$$

$$\Pi = p_1 y_1 + p_2 y_2 - vx$$

Since $\quad x = g(y_1, y_2)$

$$\Pi = p_1 y_1 + p_2 y_2 - v \cdot g(y_1, y_2)$$

$$\frac{\partial \Pi}{\partial y_1} = p_1 - v\frac{\partial g}{\partial y_1} = 0$$

$$\frac{\partial \Pi}{\partial y_2} = p_2 - v\frac{\partial g}{\partial y_2} = 0$$

Hence:

$$v = \frac{p_1}{\frac{\partial g}{\partial y_1}} = \frac{p_2}{\frac{\partial g}{\partial y_2}}$$

or

$$v = p_1\frac{\partial y_1}{\partial g} = p_2\frac{\partial y_1}{\partial g}$$

or

$$v = p_1\frac{\partial y_1}{\partial x} = p_2\frac{\partial y_2}{\partial x}$$

In other words:

MVP of x in production of y_1 must = MVP of x in production of y_2

and be equal to the price of the resource.

s.o.c.

$$- vg_{11} < 0$$

and

$$\begin{vmatrix} -vg_{11} & -vg_{12} \\ -vg_{21} & -vg_{22} \end{vmatrix} < 0$$

or

$$(-vg_{11})\ (-vg_{22}) - [(-vg_{12})\ (-vg_{21})]\ < 0$$

Ensures a PPC as follows:
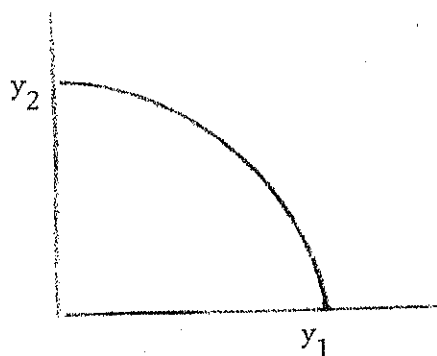
Competitive, Complementary and Supplementary Products

Competitive Production

RPT increasing in absolute value: the traditional case
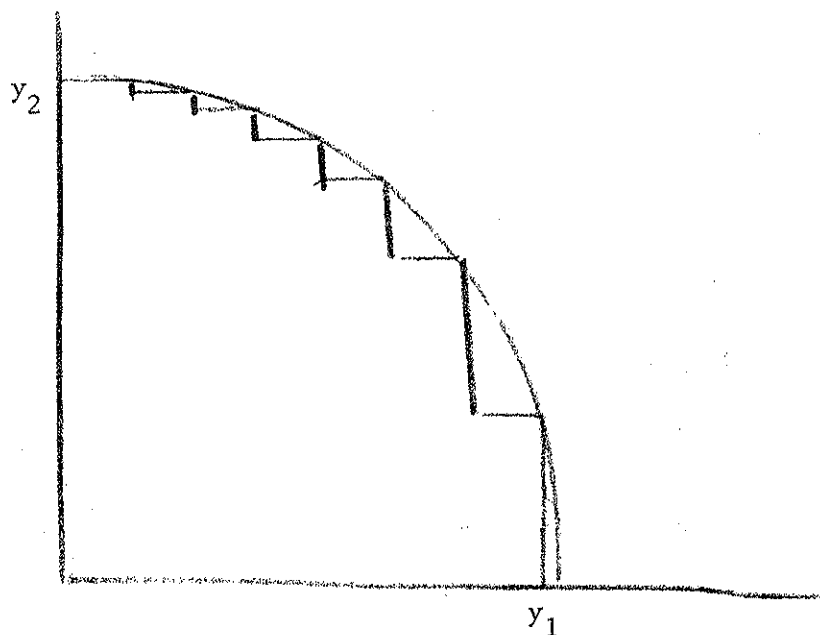
Complementary Production

Over a range, increases in production of y_2 also increase production

of y_1

(i.e., Legumes & Small Grains)
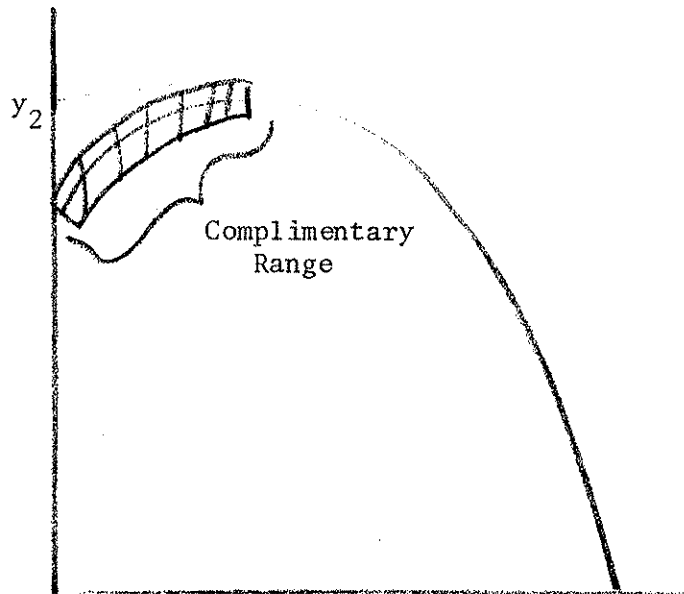
RPT increasing rather than decreasing over a range.

Supplementary Production

Over a range increases in production of y_2 do not decrease

production of y_1 (i.e., farm flock of chickens)
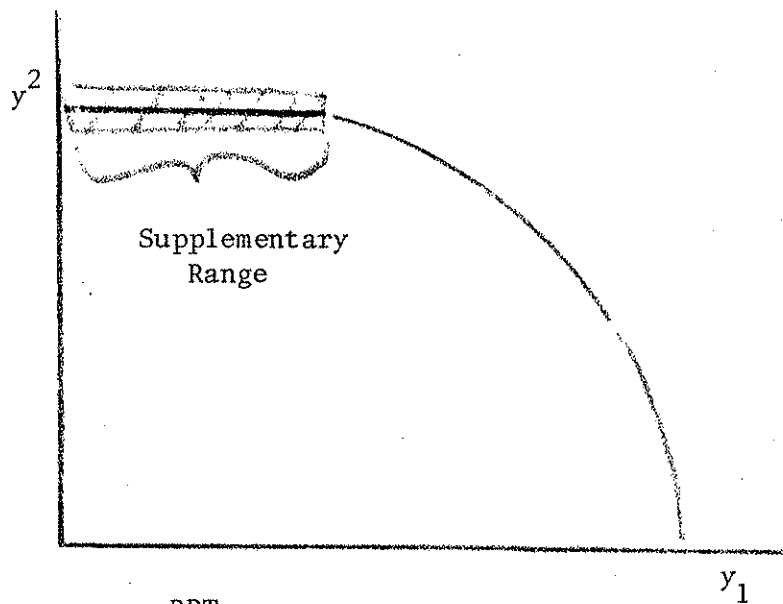
RPT = zero over a range

Some other possibilities:

Joint Products

(Beef & hides)
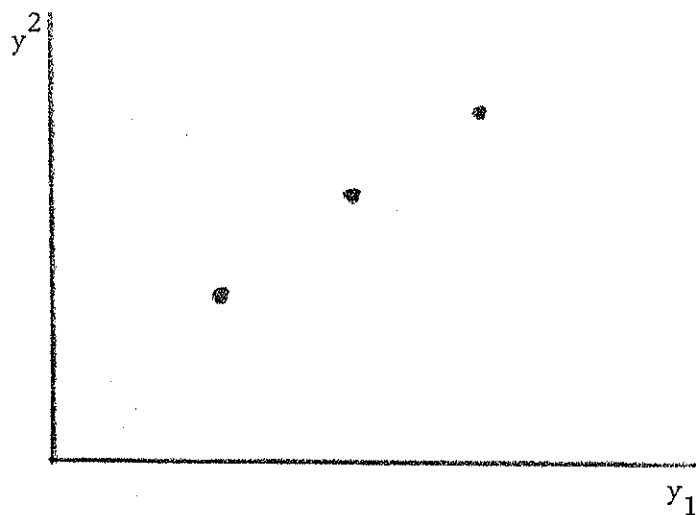
Product transformation curves consist of single points RPT

not defined.

Constant RPT
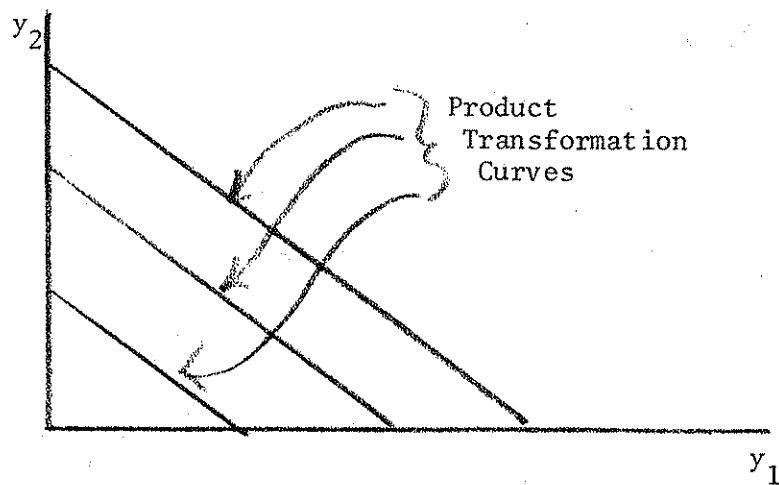
derived from 2 production functions both with constant marginal returns.

TR max - produce all y_1 if absolute slope of price ratio > slope of product

transformation curve. Produce all y_2 if absolute slope of price ratio <

slope of product transformation curve.

TR max at any point if price ratio = to RPT.

Diminishing RPT

Derived from production functions with increasing marginal returns.
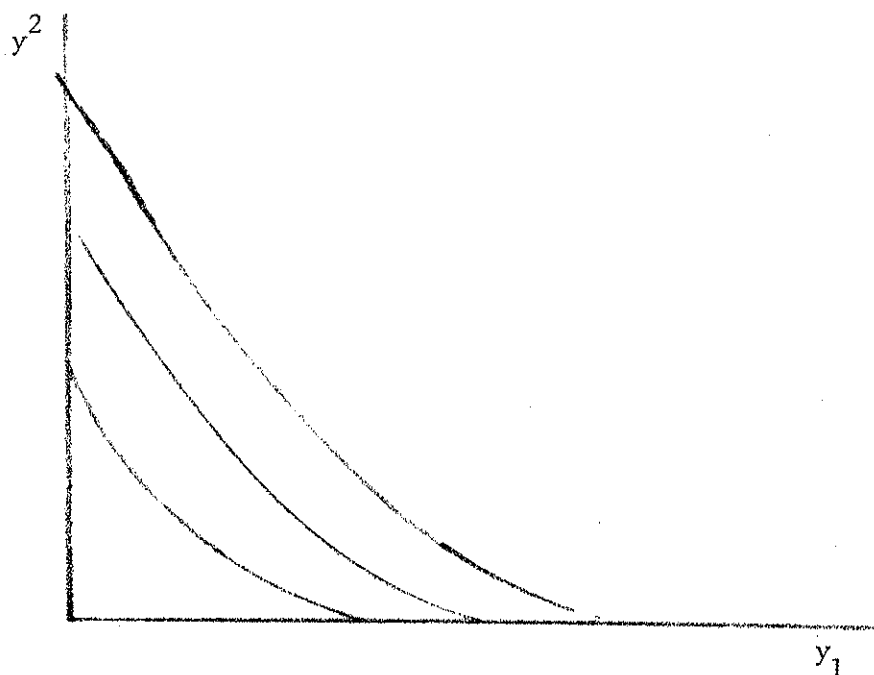
Calculus breaks down when finding TR max conditions since s.o.c. are not met (why?).
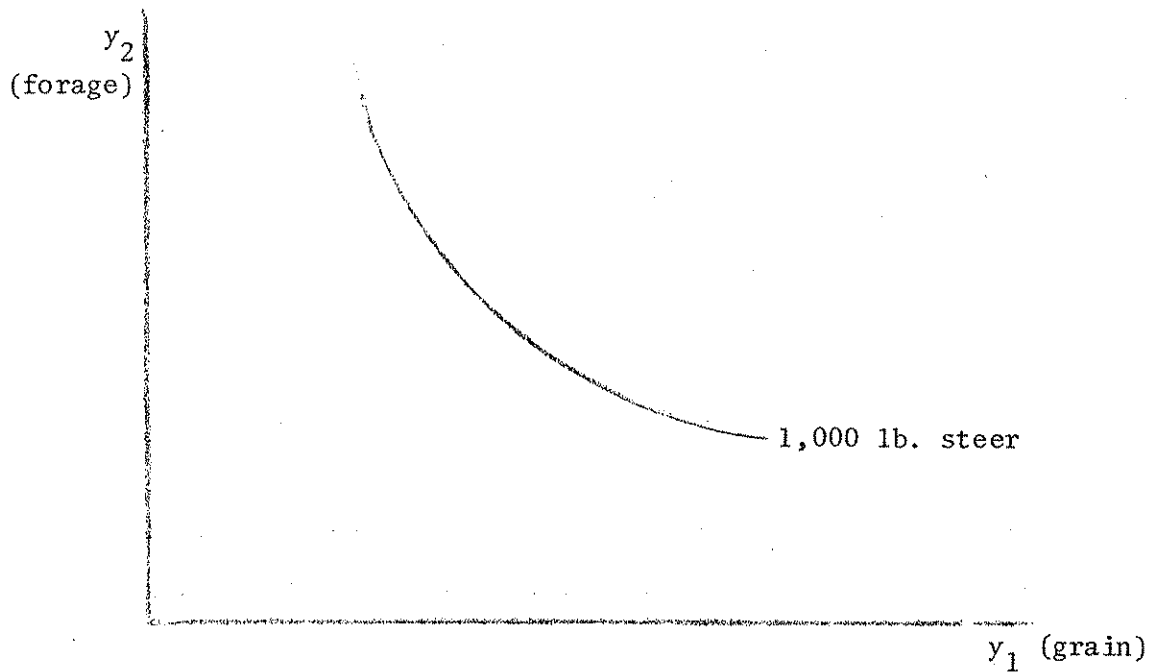
Not a very likely situation.

Combining Factor-Factor & Product-Product Models

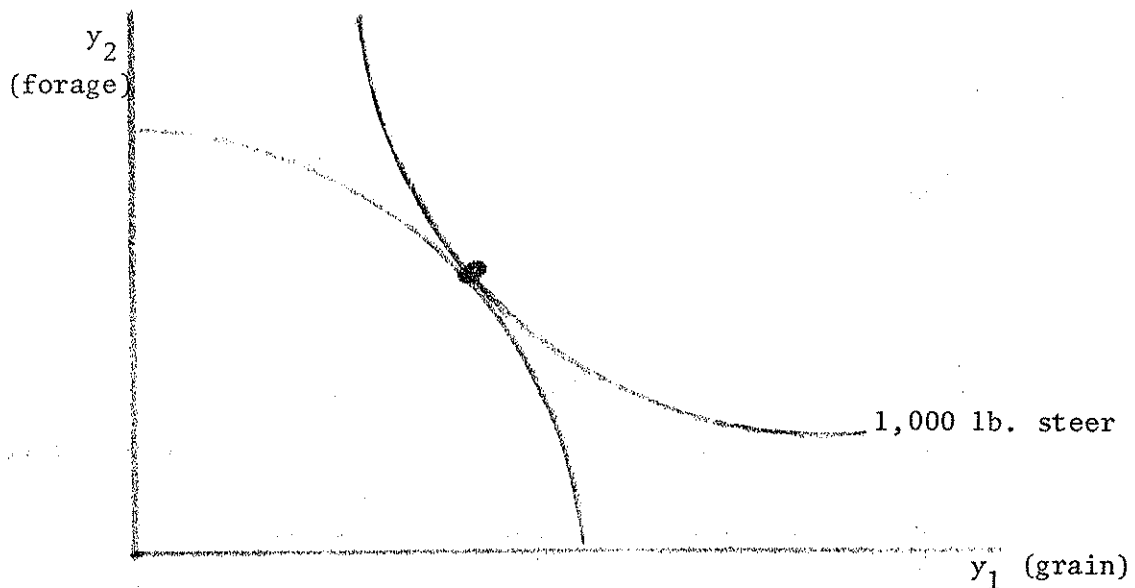Suppose: A farmer can on his farm produce 2 kinds of feed that can be used in the production of beef.

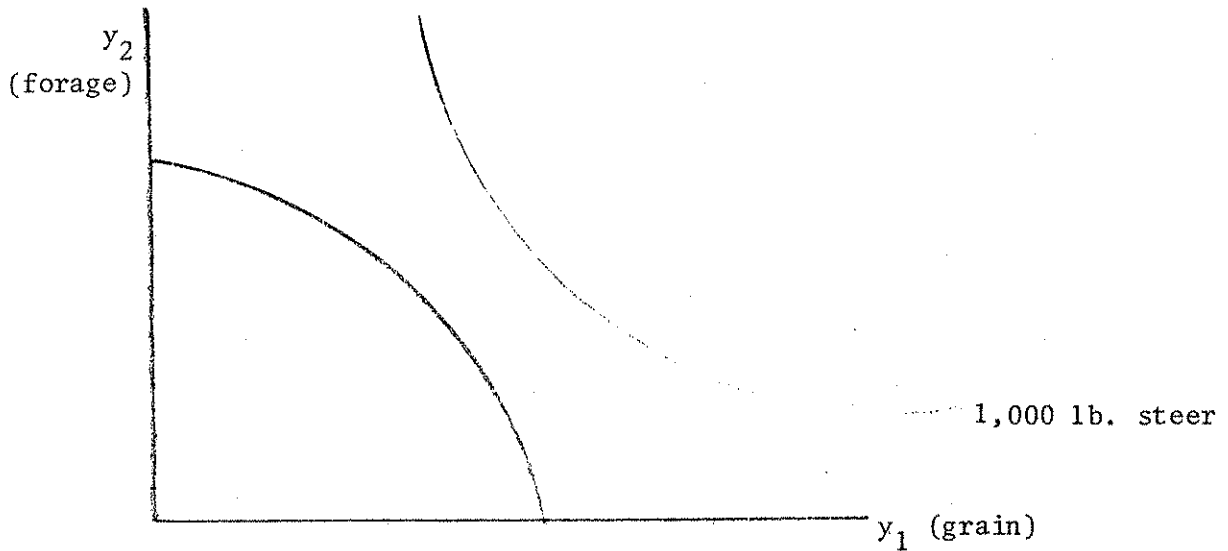P.T.C. given the fixed resources on the farm.
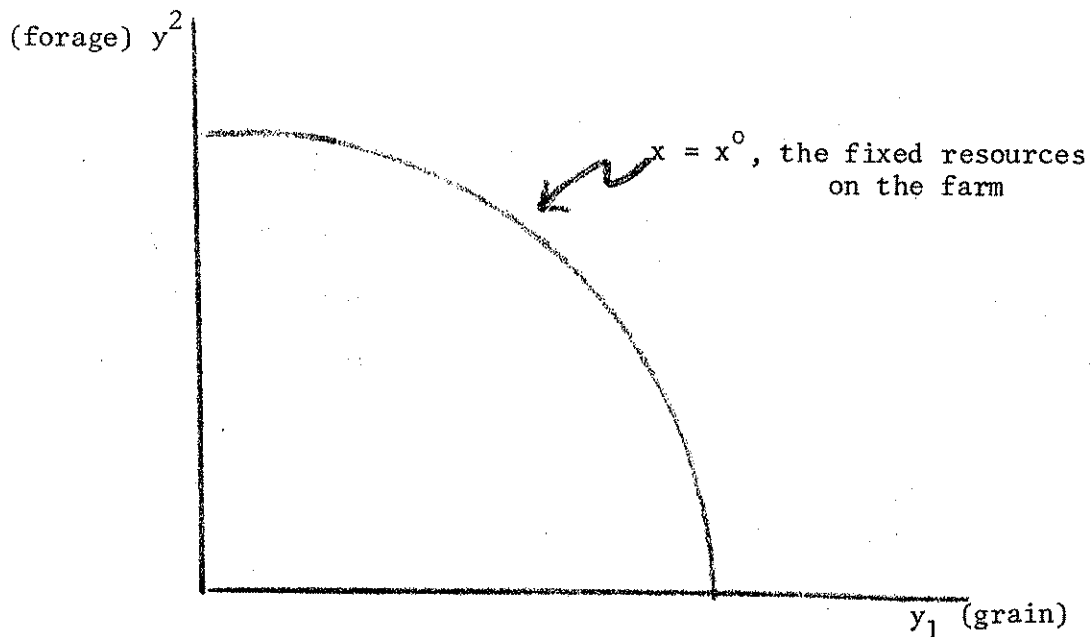
Isoquant for beef production

1,000 lb. steer

y_2 (forage)

y_1 (grain)

Superimpose: beef isoquant on product transformation curve.

1,000 lb. steer

y_2 (forage)

y_1 (grain)

Maximizes amount of beef produced where MRS of grain for forage = RPT of grain for forage.
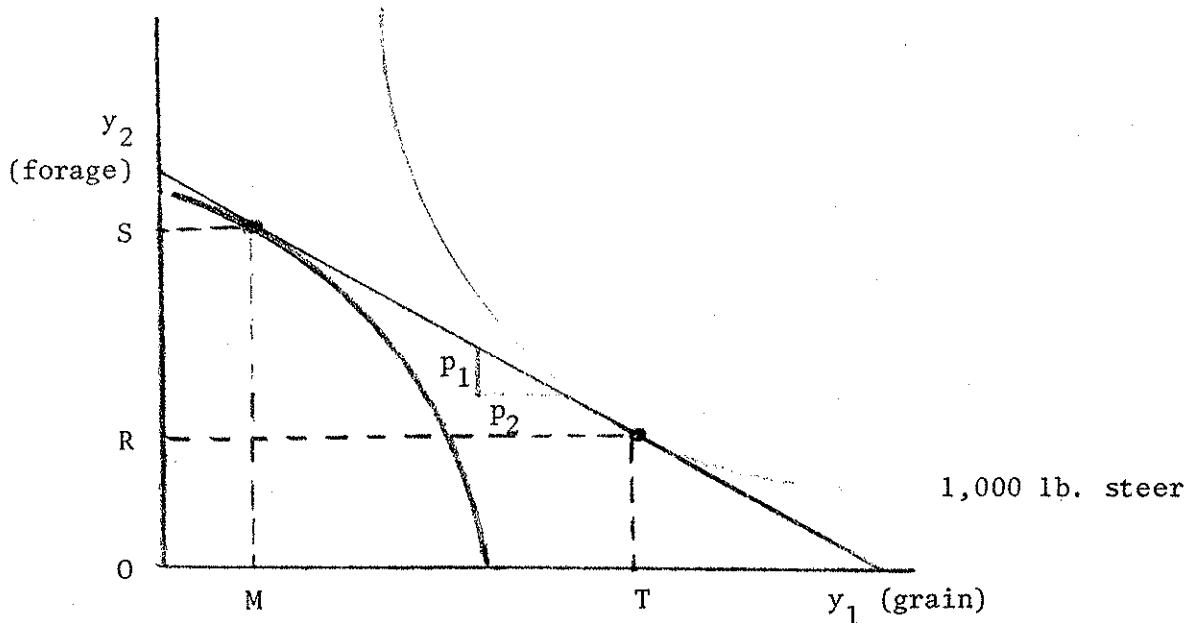
Suppose:

Is there any way for a farmer to produce a 1,000 lb. steer?

Depends on the Price ratio.

IF:

the relevant price ratio is p_1/p_2. Produce OS units of forage, OM

units of grain. Sell RS units of forage, with the money received purchase
MT units of grain.

"General Equilibrium" Model: A General Mathematical Statement of the
 Preceding:

Suppose a entrepeneur (farmer) can: (1) produce any of s possible
outputs which are valued at market prices.

Hence, he wishes to maximize revenue:

$$(1) \quad R = p_1 y_1 + p_2 y_2 + \cdots + p_s y_s$$

where y_i = the ith output and p_i is the ith price. Suppose

the production function relating inputs to outputs appears as:

$$(2) \quad \phi(y_1, y_2, \cdots y_s, x_1, x_2, \cdots x_t) = 0$$

in its implicit form; and any of t outputs can be used to produce the y_i.

Suppose the entrepeneurs cost function to be:

$$C = v_1 x_1 + v_2 x_2 \cdots + v_t x_t$$

where v_j = the price of the jth input. Formulating the lagrangian to

maximize revenue R subject to the constraint imposed by the cost function
and the production function yields:

$$(4) \quad L = p_1 y_1 + p_2 y_2 \cdots + p_s y_s$$

$$- \lambda (\phi(y_1, y_2 \cdots y_s, x_1, x_2, \cdots x_t)$$

$$- \eta (C^0 - v_1 x_1 - v_2 x_2 \cdots - v_t x_t)$$

Where:

λ and η are undetermined lagrangian multipliers.

The appropriate f.o.c. are:

$$(5) \quad \frac{\partial L}{\partial y_1} = p_1 - \lambda \frac{\partial \phi}{\partial y_1} = 0$$

$$\vdots \qquad \vdots$$

$$\frac{\partial L}{\partial y_s} = p_s - \lambda \frac{\partial \phi}{\partial y_s} = 0$$

(6) $\dfrac{\partial L}{\partial x_1} = -\lambda \dfrac{\partial \phi}{\partial x_1} + \mu v_1 = 0$

$$\vdots \qquad \vdots$$

$\dfrac{\partial L}{\partial x_t} \qquad \lambda \dfrac{\partial \phi}{\partial x_t} + \mu v_s$

(7) $\dfrac{\partial L}{\partial \lambda} = \phi(y_1,\ y_2,\ \dots\ y_s,\ x_1,\ x_2,\ \dots\ x_s)$

$\dfrac{\partial L}{\partial \mu} = (C^\circ - v_1 x_1 - v_2 x_2 - \dots - v_t x_t)$

Clearly:

(8) $\dfrac{\partial \phi}{\partial y_1} \Big/ p_1 = \dots = \dfrac{\partial \phi}{\partial y_s} \Big/ p_s = {}^1/_\lambda$

further:

(9) $\dfrac{dy_i}{dy_j} = \dfrac{p_j}{p_i} \qquad \begin{array}{l}\text{for all}\\ i \neq j\end{array}$

Moreover,

(10) $\dfrac{\partial \phi}{\partial x_1} \Big/ v_1 = \dots \dfrac{\partial \phi}{\partial x_t} \Big/ v_t$

and (11) $\dfrac{dx_i}{dx_j} = \dfrac{v_j}{v_i} \qquad \begin{array}{l}\text{for all}\\ i \neq j\end{array}$

Since:

(12) $\lambda \dfrac{\partial \phi}{\partial x_1} \Big/ v_1 = \dots = \lambda \dfrac{\partial \phi}{\partial x_t} \Big/ v_t = \mu$

and

(13) $\dfrac{\partial \phi}{\partial y_1} \Big/ p_1 = \dots = \dfrac{\partial \phi}{\partial y_s} \Big/ p_s = {}^1/_\lambda$

Then a fundamental relation holds:

$$(14) \quad \frac{p_1}{v_1} \frac{dy_1}{dx_1} = \ldots = \frac{p_1}{v_t} \frac{dy_1}{dx_t} =$$

$$\ldots = \frac{p_s}{v_1} \frac{dy_s}{dx_1} = \ldots \frac{p_s}{v_t} \frac{dy_s}{dx_t} = \mathcal{M}$$

What is the verbal interpretation of this relation?

CHAPTER VII

An Introduction to Linear Programming:

Suppose we wish to solve the following problem:

Maximize

    (1) $4x_1 + 5x_2$ (the objective function)

s.t.

    (2) $2x_1 + x_2 \leq 12$ (the constraints)

    (3) $x_1 + 2x_2 \leq 16$

Note that we make the following assumptions:

(1) the objective functions and the constraints are linear, not curve linear functions (hence the term linear programming).

(2) x_1 and x_2 are infinitely divisible - that is they do not necessarily have to assume whole numbers.

(3) Coefficients are constant and known with certainty.

Remarks:

Nonlinear programming handles situations when the objective and/or constraints are not linear functions.

Integer programming handles situations when some x's must assume whole numbers (i.e., 1 2/3 tractors or 1 5/8 bulls is not allowed).

Stochastic programming handles some situations when coefficients are known only in a probabilistic sense. There are many more kinds. *Linear programming* can hence be thought of as a subset of the general area of mathematical programming.

Note also that linear programming is not the same thing as computer programming, although most linear programming algorithms are solved with the aid of a computer.

The graphical illustration of the linear programming problem reveals that the 2 constraints form a "quasi" production possibilities curve. Note that only the shaded area interior to both constraints is relevant. (In particular, let Set A denote the area bounded by equation (2); Set B denote the area bounded by equation (3). Then our feasible set (shaded area bounded by the PPC) is denoted by $A \cap B$.)

How are the slopes of the objective function and constraint functions obtained?

Why is the area exterior to $A \cap B$ irrelevant?

Note also that the "optimum solution" to the LP is that point on the PPC which moves the objective function furthest to the NE.

OBJECTIVE FUNCTION

No longer relevant

No longer relevant

Feasible Solution Area

PRIMAL

x_1

x_0
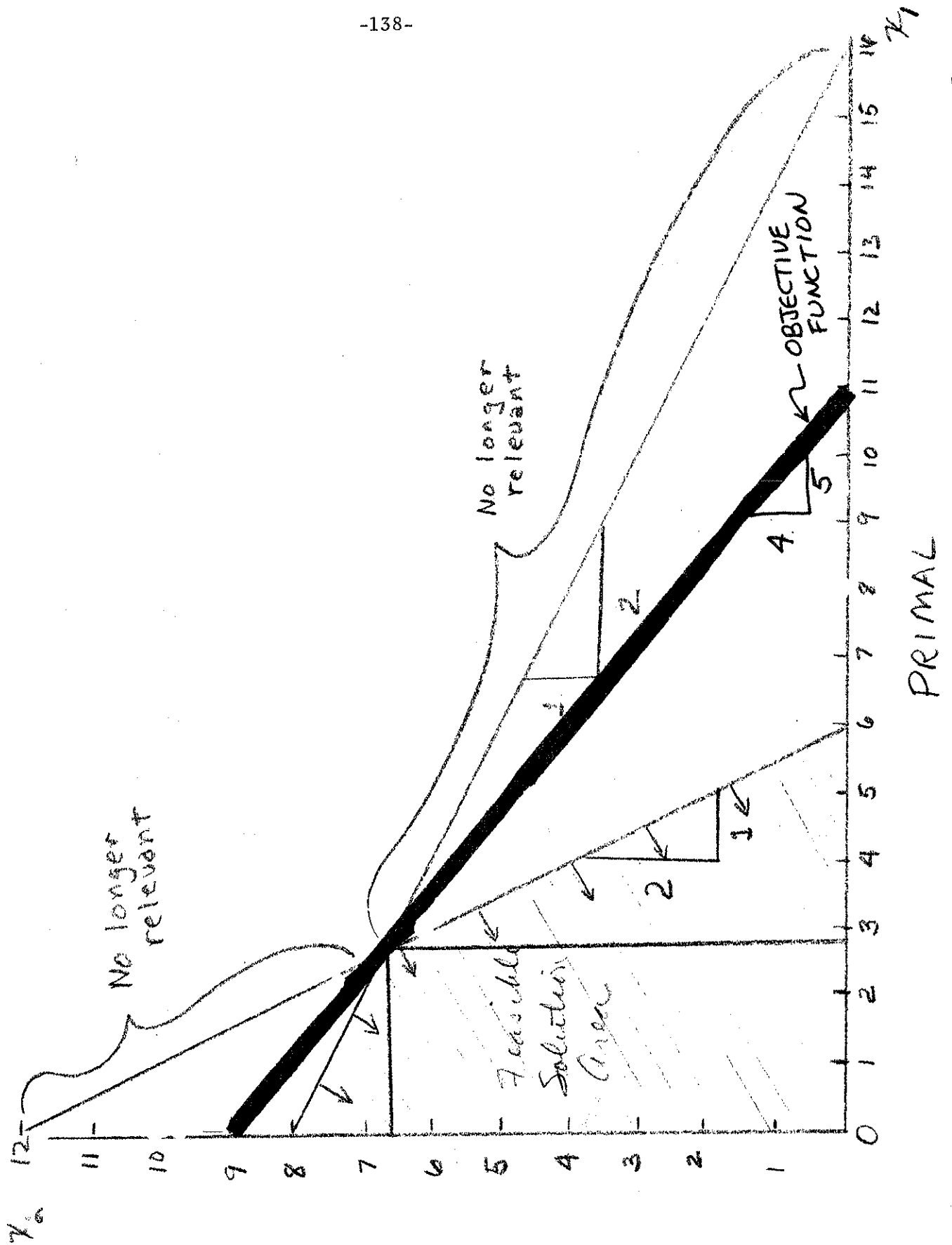
Introduction to the Simplex

Graphical solutions to linear programming problems are cumbersome and impractical for problems with more than 2 constraints.

One method for solving an LP without graphs is the Simplex Method. The Kuhn Tucker conditions outlined earlier are at the heart of the solution method.

We begin with the original equations

$$4x_1 + 5x_2 \quad \text{(a max.)}$$

s.t.

$$2x_1 + x_2 \leq 12$$

$$x_1 + 2x_2 \leq 16$$

Now add positive slacks to each equation

$$u_1$$

$$u_2$$

Hence, our problem becomes

$$4x_1 + 5x_2 \quad \text{(a max.)}$$

s.t.

$$2x_1 + x_2 + 1u_1 + 0u_2 = 12$$

$$x_1 + 2x_2 + 0u_1 + 1u_2 = 16$$

Note that:

(1) $u_1, u_2 \geq 0$

(2) u_1 appears in equation 1,

u_2 appears in equation 2,

u_2 has a coefficient of zero in equation 1

u_1 has a coefficient of zero in equation 2.

or

$$\begin{bmatrix} 2 & 1 & 1 & 0 & 12 \\ 1 & 2 & 0 & 1 & 16 \\ 4 & 5 & 0 & 0 & 0 \end{bmatrix}$$

Note that the 12 and 16 appear on the right-hand side of the matrix. Hence, they are known as the right-hand side (RHS). Sometimes they appear on the left-hand side of the matrix. This really doesn't make any difference in solving the problem, but people have become so used to calling them the right-hand side, they call them the right-hand side even if they are on the left-hand side. So much for right-hand sides.

The Simplex procedure is nothing more than a series of matrix operations designed to sweep out the original coefficient matrix converting it to an identity matrix.

	Column				
Row	(1)	(2)	(3)	(4)	(5)
(1)	2	①	1	0	12
(2) →	1	②*	0	1	16
(3)	4	5↑	0	0	0

The ↑ indicates the entering column.

(2) This is chosen because x_2 contributes more to profit than x_1.

Actually we can choose either.

Now, since $\dfrac{16}{②} = 8$

$\dfrac{12}{①} = 12$

The second resource is the most limiting.

The $\rightarrow$ indicates the entering row since the second resource is most limiting.

② * becomes the pivotal element.

We begin by dividing every value in row (2) by ② * the pivotal element forming a new row (2).

(N2)   1/2   1   0   1/2   8

Now take old row (1), and subtract from it the new row (N2) 2 times ① which is the intersection between the entering column and the old row 1.

In short:

(N1) = (1) - ① x (N2)

(N1) = 2-1/2 x ①   1-1 x ①   1-0 x ①   0-1/2 x ①   12-8 ①

= 1 1/2   0   1   - 1/2   4

The new profit row is

(N3) = 4-1/2 x ⑤   5-1 x ⑤   0-0 x ⑤   0-1/2 x ⑤   0-8 x ⑤

1 1/2   0   0   -2.50   -40

Hence, the new matrix is:

Row	Column				
	(1)	(2)	(3)	(4)	(5)
(N1)	→ (1 1/2) *	0	1	- 1/2	4
(N2)	(1/2)	1	0	1/2	8
(N3)	1 1/2 ↑	0	0	-2.50	-40

The 1 1/2 in N3 indicates that some additional x_1 would be profitable.

$$\frac{4}{(1\ 1/2)} = \frac{8}{3}$$

$$\frac{8}{(1/2)} = 16$$

Following the same procedure, the new matrix becomes:

(NN1)	1	0	2/3	- 1/3	8/3
(NN2)	0	1	- 2/6	2/3	6 2/3
(NN3)	0	0	$ - 1	$ - 2	$ - 44

The solution values are 8/3 units of x_1 and 6 2/3 units x_2. The maximum value of the objective function is \$44. The Implicit worth (shadow price) of x_1 is \$1. The Implicit worth (shadow price of x_2 is \$2).

Link to Kuhn Tucker Conditions

The same problem can be solved using lagrangian multipliers.

Maximize:

$$4x_1 + 5x_2$$

s.t.

$$2x_1 + x_2 \leq 12$$

$$x_1 + 2x_2 \leq 16$$

Add positive slacks

$$2x_1 + x_2 + 1u_1 + 0u_2 = 12$$

$$x_1 + 2x_2 + 0u_1 + 1u_2 = 16$$

Formulate the Lagrangian:

$$L = 4x_1 + 5x_2 - \lambda_1(2x_1 + x_2 + u_1 - 12)$$

$$- \lambda_2(x_1 + 2x_2 + u_2 - 16)$$

Unknowns are x_1, x_2, u_1, u_2, λ_1 and λ_2

Maximizing f.o.c.

(1) $\dfrac{\partial L}{\partial x_1} = 4 - 2\lambda_1 - \lambda_2 = 0$

(2) $\dfrac{\partial L}{\partial x_2} = 5 - \lambda_1 - 2\lambda_2 = 0$

(3) $\dfrac{\partial L}{\partial u_1} = -\lambda_1 = 0$

(4) $\dfrac{\partial L}{\partial u_2} = -\lambda_2 = 0$

(5) $\quad\dfrac{\partial L}{\partial \lambda_1} = 2x_1 + x_2 + u_1 - 12 = 0$

(6) $\quad\dfrac{\partial L}{\partial \lambda_2} = x_1 + 2x_2 + u_2 - 16 = 0$

but

(7) $\quad u_1 = 12 - 2x_1 - x_2$

(8) $\quad u_2 = 16 - x_1 - 2x_2$

Hence, (3) & (4) become

(9) $\quad -\lambda_1 \cdot u_1 = 0$

(10) $\quad -\lambda_2 \cdot u_2 = 0$

(11) $\quad -\lambda_1 (12 - 2x_1 - x_2) = 0$

(12) $\quad -\lambda_2 (16 - x_1 - 2x_2) = 0$

Rearranging (1) and (2)

(13) $\quad \lambda_2 = 4 - 2\lambda_1$

(14) $\quad \lambda_1 = 5 - 2\lambda_2$

(15) $\quad \lambda_2 = 4 - 2(5 - 2\lambda_2)$

(16) $\quad \lambda_2 = 4 - 10 + 4\lambda_2$

(17) $\quad -3\lambda_2 = -6$

(18) $\quad \lambda_2 = 2 \quad$ Shadow price for x_2

(20) $\lambda_1 = 1$ Shadow price for x_1

Inserting these values in (11) and (12)

(21) $-1 \ (12 - 2x_1 - x_2 = 0)$

$2x_1 + x_2 - 12 = 0$

(22) $-2 \ (16 - x_1 - 2x_2) = 0$

(23) $-2x_1 + 4x_2 - 32 = 0$

(24) $x_1 = 16 - 2x_2$

(25) $2(16 - 2x_2) + x_2 - 12 = 0$

(26) $32 - 4x_2 + x_2 - 12 = 0$

(27) $-3x_2 = -20$

(28) $x_2 = \dfrac{20}{3} = 6 \ 2/3$

(29) $x_1 = 16 - 2(6 \ 2/3) = 2 \ 2/3$ Optimum values for x_1 and x_2

The Dual

Recall that in our work with Lagrangian constrained optimization problems, the solution that maximized output subject to the budget constraint was the same solution that minimized cost subject to the output constraint. The same analogy exists in a linear programming context.

Our original (Primal) problem was:

$$4x_1 + 5x_2 \quad \text{(a max.)}$$

s.t.

$$2x_1 + x_2 \leq 12$$

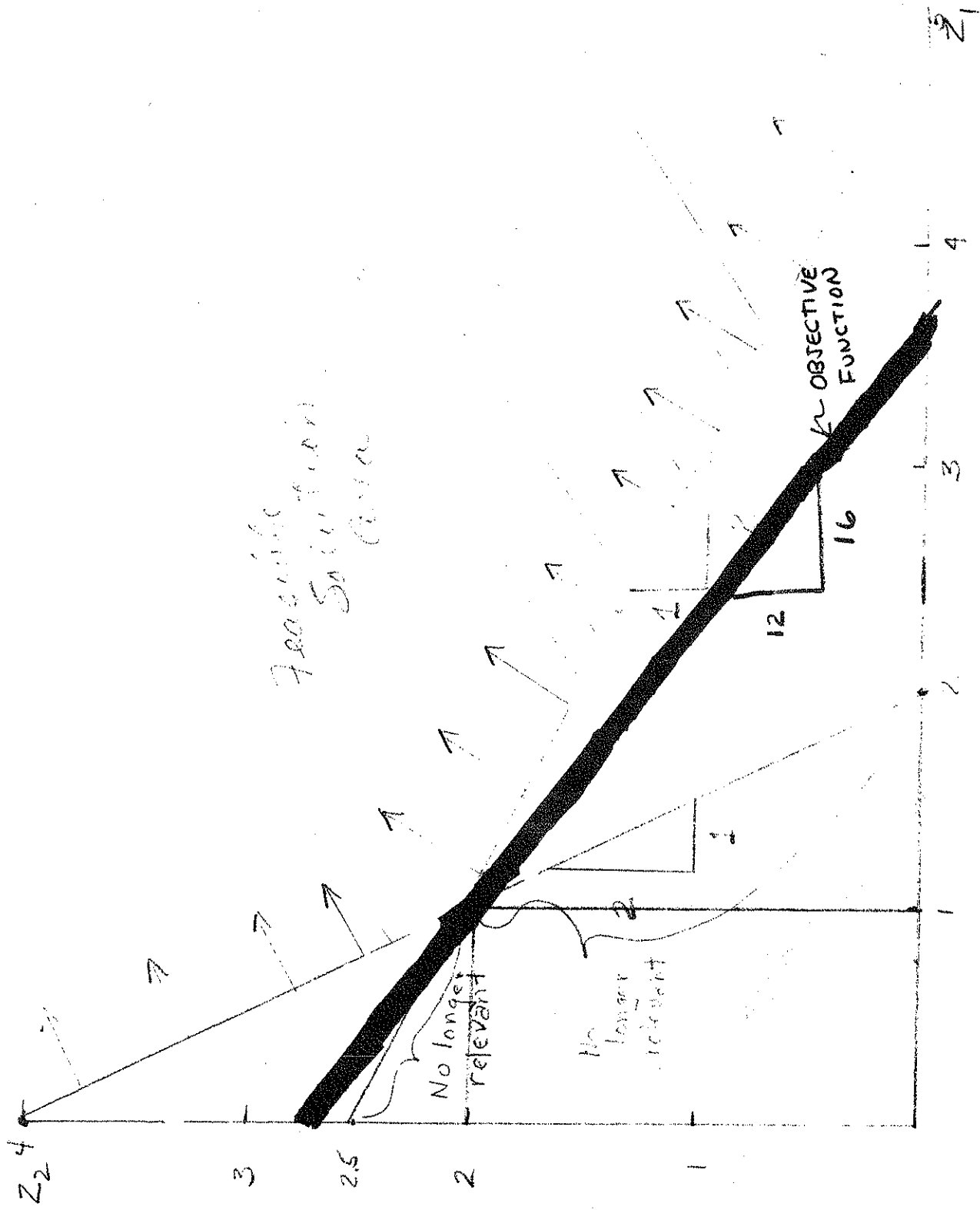$$x_1 + 2x_2 \leq 16$$

The corresponding dual is:

$$12z_1 + 16z_2 \quad \text{(a min.)}$$

s.t.

$$2z_1 + z_2 \geq 4$$

$$z_1 + 2z_2 \geq 5$$

Note that:

(1) In the primal is a maximization problem, the dual is a minimization problem, and *vice versa*.

(2) The inequality constraints are reversed.

(3) The new unknowns are z_1 and z_2.

The rows of the dual essentially correspond to the columns of the primal and *vice versa*.

Note from the graph that the feasible solution area is bounded from below by portions of the 2 constraints. These portions of constraints form an (quasi) isoquant with linear segments, just as with the primal. The constraints formed a (quasi) PTC or PPC.

Note also that the solution values for z_1 and z_2 in the dual are the shadow prices (values of lagrangian multipliers λ_1 and λ_2) in the primal.

The dual is solved via the simplex method following the same procedure as the primal. The simplest way of solving via the simplex entails merely multiplying the objective function and constraints by (-1) making the new problem a maximization and recalling that an inequality is reversed when an equation is multiplied by -1.

We solve

$$-12z_1 - 16z_2 \quad \text{(a max.)}$$

s.t.

$$-2z_1 - z_2 \leq -4$$

$$-z_1 - 2z_2 \leq -5$$

We can also solve the dual directly with a few simple procedural changes. Exercise:

(1) Solve the above revised dual.

We can also show that the lagrangian multipliers in the dual are the solution values for x_1 and x_2 in the primal.

Exercise:

(1) Solve the dual following the Kuhn Tucker conditions.

CHAPTER VIII

Market Equilibrium

Demand Functions

The demand by the ith consumer for the jth commodity produced by a producer can be expressed as:

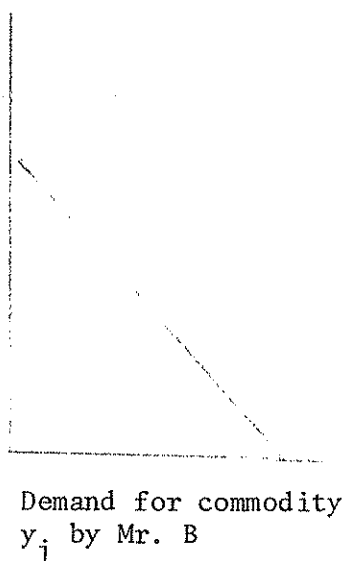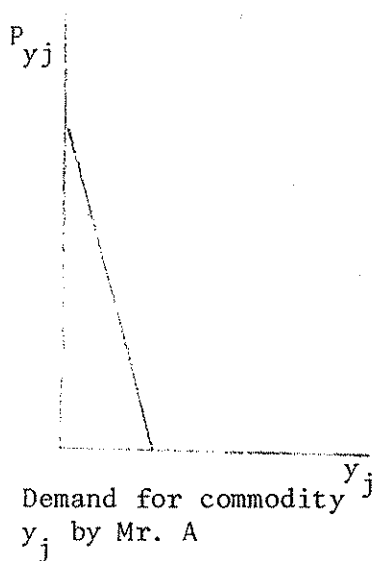$$y_{ij} = \phi(p_1, p_2 \cdots p_j, \cdots p_n, I_i)$$

where

$p_1, \cdots p_n$ is a vector of prices for the n commodities

I_i = the consumers (real) income

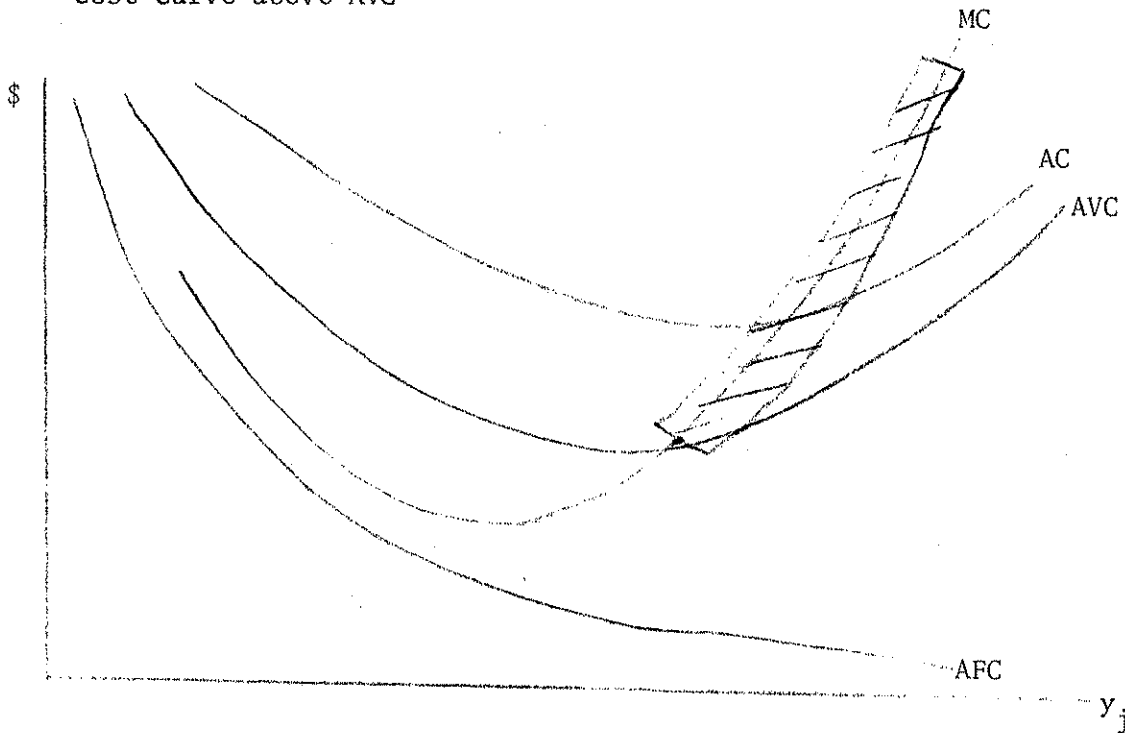Aggregate Demand for all m consumers for the jth commodity

$$\sum_{i=1}^{m} y_{ij} = \Sigma \phi(p_1, p_2 \cdots p_j, \cdots p_n, I_i)$$

In short:

Demand for commodity
y_j by Mr. A

Demand for commodity
y_j by Mr. B

Aggregate
Demand

Supply functions

The individual producers short run supply curve for y_y is his marginal cost curve above AVC

Why?

In the long run all costs are variable (Why?) and the Supply Curve for the firm is the long run marginal cost function.

Exercise:

Suppose

$$TC = .2 \, y_j^3 - 2 \, y_j^2 + 18 \, y_j + 12$$

(1) Find MC,

(2) Find the supply curve for the firm expressed as a function of price.

(3) Find AVC.

(4) Find the aggregate supply if the industry consists of 50 identical firms.

Models of Imperfect Competition

Key assumption:

A downsloping demand curve

$$\frac{dy}{dp} < 0$$

In other words:

$$y = f(p)$$
$$f'(p) < 0$$

Monopolist – the firm is the industry. Hence, the demand curve for the product of a monopolist is similar to the demand curve for an industry.
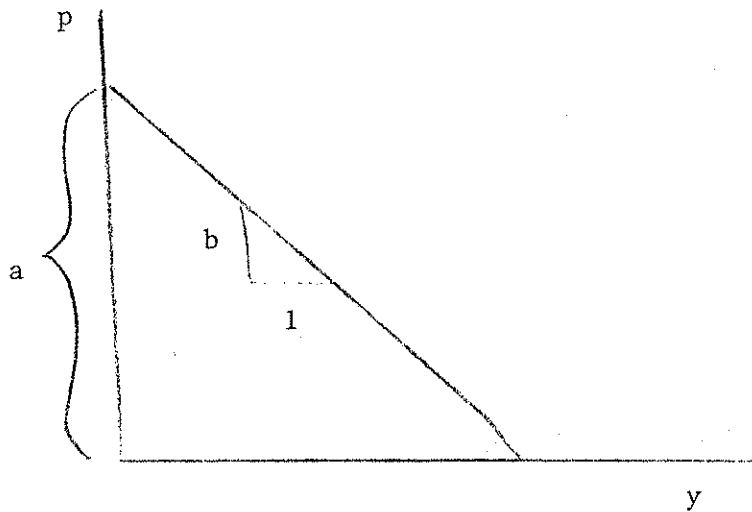
A monopolists total revenue is:

$$TR = p \cdot y$$

Marginal revenue is:

$$\frac{dTR}{dy} = p + y \frac{dp}{dy}$$

Since $\dfrac{dp}{dy} < 0$, MR always lies below price when a downsloping demand curve exists. MR declines at a twice the rate of the demand curve. Let the demand curve be: $p = a - by$
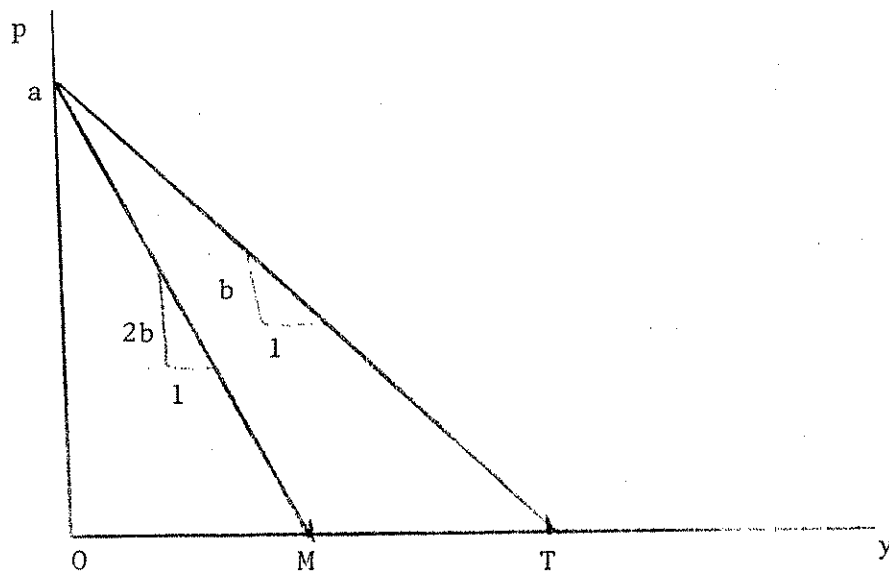

Since:

$$TR = py$$

Hence:

$$TR = ay - by^2$$

$$MR = a - 2by$$

$$\frac{OM}{MT} = 1$$

Furthermore:

$$TR = \int_o^{y'} (a-2by)\ dy$$

Elasticity of Demand

$$e = -\frac{p}{y} \cdot \frac{y}{p}$$

Since:

$$MR = p + y\ \frac{dp}{dy}$$

$$MR = p(1 + \frac{y}{p}\ \frac{dp}{dy})$$

$$= p(1 - \frac{1}{e})$$

$$MR < 0 \quad if \quad |e| < 1$$

$$MR = 0 \quad if \quad |e| = 1$$

$$MR > 0 \quad if \quad |e| > 1$$

Profit Maximization

$$TR = f(y)$$

$$TC = g(y)$$

$$\Pi\ = TR - TC$$

$$\frac{d\Pi}{dy} = \frac{df}{dy} - \frac{dg}{dy} = 0$$

$$MR = MC$$

$$\frac{d^2\Pi}{dy^2} = \frac{d^2f}{dy^2} - \frac{d^2g}{dy^2} < 0$$

$$\frac{d^2f}{dy^2} < \frac{d^2g}{dy^2}$$

The slope of MR must be < the slope of MC. If MR is decreasing and MC is increasing, this condition is satisfied.

Some Applications:

Discriminating Monopolist

Suppose a market can be segmented.

    Demand for oranges.
                fresh market.
                juice market.

Even though oranges for both markets may be identical, it is probably more profitable to charge more for fresh oranges than for juice oranges.

    Let $y = y_1 + y_2$

    where y = total available oranges

            y_1 = amount sold in first market

            y_2 = amount sold in second market

    Then:

$$\Pi = TR_1(y_1) + TR_2(y_2)$$

$$- TC(y_1 + y_2)$$

$$\frac{\partial \Pi}{\partial y_1} = \frac{\partial TR}{\partial y_1} - \frac{\partial TC}{\partial y_1} = 0$$

$$\frac{\partial \Pi}{\partial y_2} = \frac{\partial TR}{\partial y_2} \quad \frac{\partial TC}{\partial y_2} = 0$$

Since by assumption it costs the same to produce an orange for either market

$$\frac{\partial TC}{\partial y_1} = \frac{\partial TC}{\partial y_2} = MC_y$$
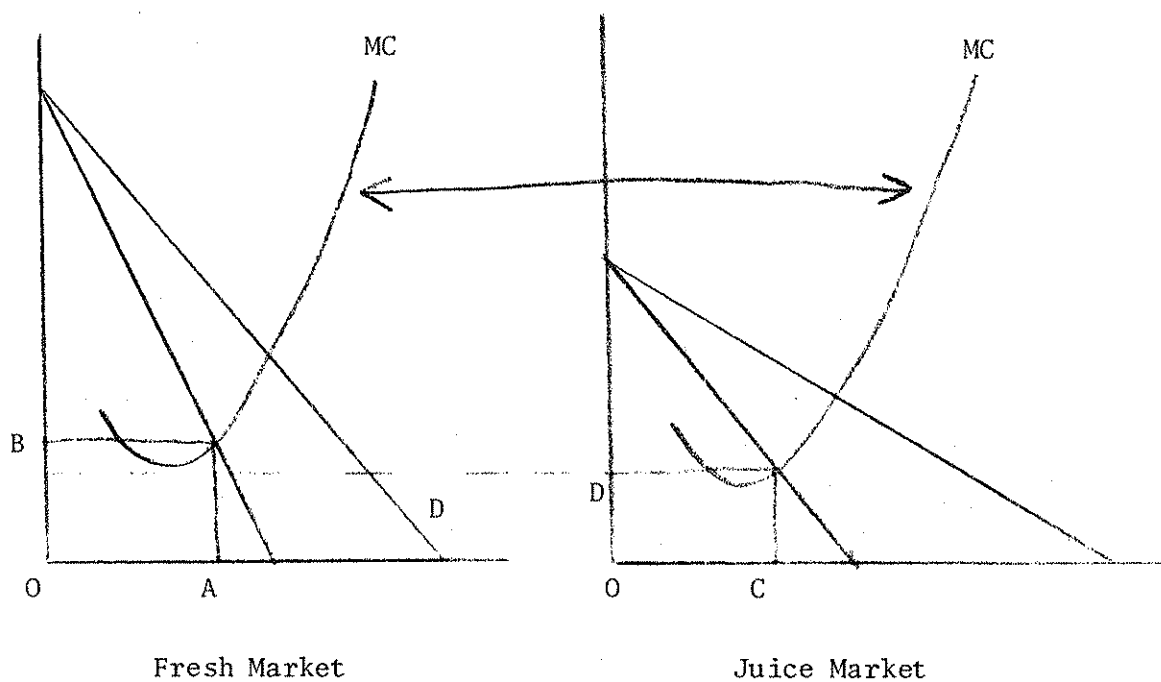
Hence:

$$\frac{\partial TR}{\partial y_1} = \frac{\partial TR}{\partial y_2} = MC_y$$

$$MR_{y_1} = MR_{y_2} = MC_y$$

The Marginal Revenue in each market must equal the Marginal Cost of the output as a whole.

A picture:

Fresh Market   Juice Market

Entrepreneur produces OA + OC units of oranges -- sells oranges for OB in the fresh market and OD in the juice market. Note also that if it is to be profitable to segment markets, the elasticity of demand in each market must be different.

The Perfectly Discriminating Market and the Extraction of Consumer Surplus

The perfectly discriminating monopolist is able to extract all the consumer surplus by selling each unit of output to the individual who pays the highest price.

A picture:
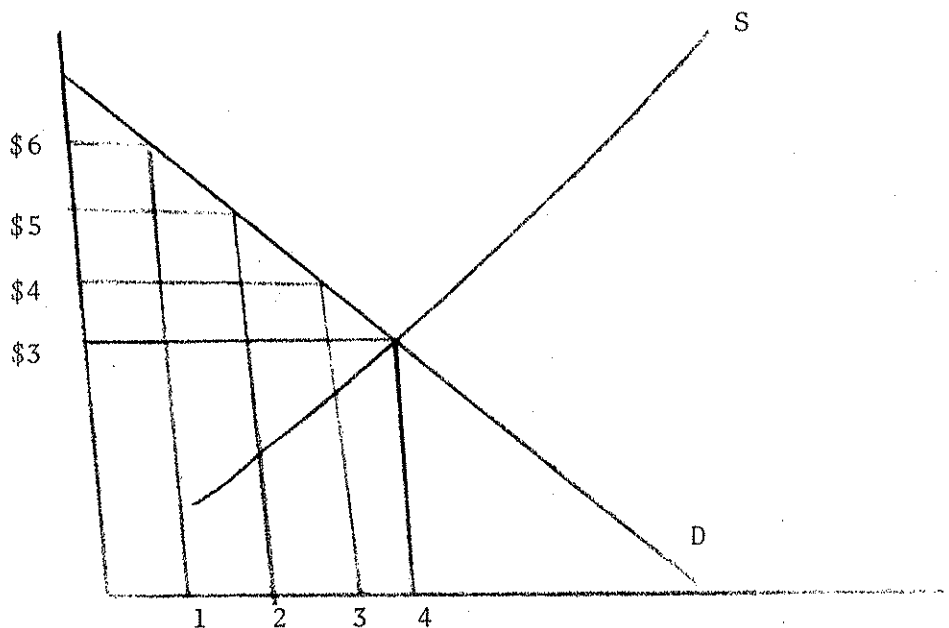
All purchasers do not pay $3 for the output. The first purchaser is willing to and does pay $6, the second purchaser $5 and so forth. Total revenue to the monopolist in this example is:

$$\$6 \cdot 1 + \$5 \cdot 1 + \$4 \cdot 1 + \$3 \cdot 1 = \$18$$

Ordinarily, total revenue would be:

$$4 \times \$3 \text{ or } \$12$$

Problem -- how do you discriminate among purchasers, if each unit of the commodity is the same.

There have been some experiments with the Dutch Auction for selling livestock. Start bidding at extremely high price. Price is gradually lowered. This is an attempt to extract the consumer surplus we have described. Of course, under these conditions
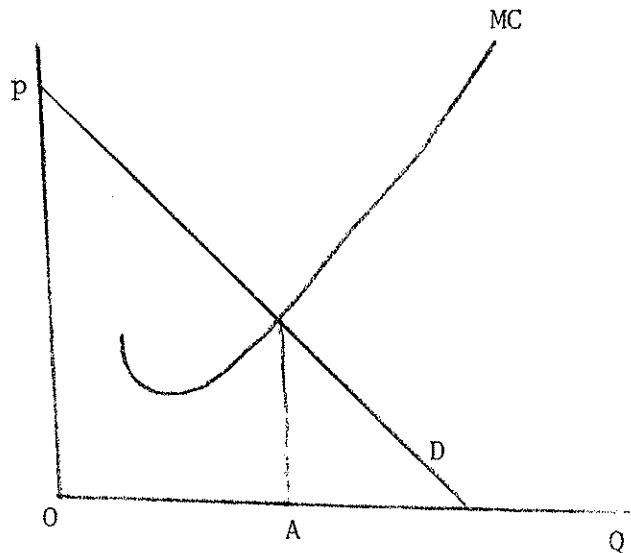
$$D = D(y)$$

(Demand for output y)

$$\Pi = \int_0^{y_0} D(y) \, dy - TC(y)$$

Then:

$$\frac{d\Pi}{dy} = D(y) - \frac{dTC}{dy} = 0$$

$$\frac{d\Pi}{dy} = D = MC$$

Profits are maximized where:

Entrepreneur produces OA units of output. Why would the entrepreneur not want to produce more than OA units of output?

The price of each unit sold, of course, would vary.

Mathematical Concepts
Basic to Production Economics

1. Find $\frac{dy}{dx}$ for the following functions:

     i $y = 2x$

     ii $y = x^2$

     iii $y = x/3$

     iv $y = f(x)$

     v $y = 2ax$

2. Integrate the following function:

$$\int (x^2 + 3x^3)\ dx =$$

3. Maximize:

$$x^3 + 4x^2$$

4. Maximize:

$$u = u(x_1,\ x_2)$$

s.t.

$$I = p_1 x_1 + p_2 x_2$$

5. Find the total differential of the equation:

$$y = f(x_1, x_2)$$

6. Evaluate the following definite integral

$$\int_2^4 x^2 \, dx$$

Intermediate Production Economics

1. What do the following abbreviations stand for:

    (a) MC

    (b) MR

    (c) MRC

    (d) MPP

    (e) TPP

    (f) MVP

    (g) TR

    (h) TC

    (i) AVC

    (j) AC

    (k) TVC

2. Draw the traditional 3 stage single input-single output production
    function. Show the stages of production, APP, MPP, MRC, and any
    other details you think are worthwhile.