Non-Parametric Estimation of Marginal Productivity, Returns to Scale and Production Elasticity†

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In this paper, vector algebra is utilised to derive estimates of marginal productivity, returns to scale and production elasticity without parametric specification or estimation of production functions. The approach developed here does not require any but the assumption of cost minimisation. Applications of the proposed framework to the US and Japanese manufacturing data yield results which are consistent with normal expectations.

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1 Introduction

Estimating marginal product, returns to scale and production elasticity is important in empirical studies of production theory. Specifically, isolating scale effect is crucial in order to obtain unbiased estimates of technical change (Stigler 1961, Wan and Jia 1992). However, marginal product, scale and production elasticities have conventionally been obtained *via* specification and estimation of a parametric production function. Apart from the many usual problems of econometric model specification and estimation, the estimates obtained are function-dependent and thus sensitive to the hypotheses maintained in the particular production function being postulated.

The ideal approach is to obtain these estimates without imposing <u>any</u> assumptions on the underlying technology (e.g., homotheticity, constant marginal rate of technical substitution). This is exactly what the present note is aimed at. Both the approach used and the calculations involved in this paper are simple.

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2 Non-Parametric Estima*ion

Defining the following notations: $X_t = (x_{t1}, x_{t2}, \cdots, x_{tN})$ denotes the observed input vector in period t with corresponding prices $P_t = (p_{t1}, p_{t2}, \cdots, p_{tn})$; Y_t = the observed output in period t; f = the true but unknown production function transforming X_t into Y_t. Thus, $Y_t = f(X_t)$ and $Y_{t+1} = f(X_{t+1})$. Now, consider two input vectors, say X_t and X_{t+1}, in the N-dimensional input space and let $\Delta X_t = X_{t+1} - X_t$ with its direction being represented by l. By definition, the directional derivative of the underlying production function at X_t along the direction l is

$$\begin{aligned} \frac{\partial f}{\partial l} &= \sum_{i=1}^{N} \frac{\partial f}{\partial x_{ti}} \cos \alpha_i \\ &= \sum_{i=1}^{N} \frac{\partial f}{\partial x_{ti}} \frac{\Delta x_{ti}}{|\Delta X_t|}, \end{aligned} \tag{1}$$

where $\cos \alpha_i = \Delta x_{ti}/|\Delta X_t|$ is the i-th directional cosine of the vector ΔX_t. Given observations on Y_t, Y_{t+1}, X_t and X_{t+1}, the derivative can be evaluated as

$$\begin{aligned} \frac{\partial f}{\partial l} &\stackrel{def}{=} \lim_{|\Delta X_t| \to 0} \frac{f(X_{t+1}) - f(X_t)}{|\Delta X_t|} \\ &\approx \frac{Y_{t+1} - Y_t}{|\Delta X_t|} \\ &= \frac{\Delta Y_t}{|\Delta X_t|}, \end{aligned} \tag{2}$$

where $\Delta Y_t = Y_{t+1} - Y_t$. Under cost minimisation (an assumption commonly made, particularly in duality theory and in the studies of technical change), the following is true:

$$\frac{\partial f / \partial x_{ti}}{\partial f / \partial x_{tj}} = \frac{p_{ti}}{p_{tj}}, \tag{3}$$

where p_{tr} is the r-th input price prevailing at time t.

By definition, the gradient vector of f at X_t, denoted by ∇f, consists of partial derivatives $\partial f / \partial x_{ti}$ as its elements, thus using (3), it is easy to show that

$$\frac{\partial f/\partial x_{ti}}{|\nabla f|} = \frac{p_{ti}}{\sqrt{\sum_{j=1}^{N} p_{tj}^2}}, \tag{4}$$

so,

$$\frac{\partial f}{\partial x_{ti}} = |\nabla f| \frac{p_{ti}}{\sqrt{\sum_{j=1}^{N} p_{tj}^2}}. \tag{5}$$

Combining equations (1), (2) and (5), we have

$$\begin{aligned}\frac{\partial f}{\partial l} &= \sum_{i=1}^{N} |\nabla f| \frac{p_{ti}}{\sqrt{\sum_{j=1}^{N} p_{tj}^2}} \frac{\Delta x_{ti}}{|\Delta X_t|} \\ &\approx \frac{\Delta Y_t}{|\Delta X_t|}.\end{aligned}$$

That is,

$$\frac{|\nabla f|}{\sqrt{\sum_{j=1}^{N} p_{tj}^2}} \approx \frac{\Delta Y_t}{\sum_{i=1}^{N} p_{ti} \Delta x_{ti}}.$$

Substituting the above into (5) yields

$$\frac{\partial f}{\partial x_{ti}} \approx \frac{\Delta Y_t p_{ti}}{\sum_{j=1}^{N} p_{tj} \Delta x_{tj}}. \tag{6}$$

The above formula can be used for calculating marginal productivities. It is interesting to note that if the assumption that factors were paid their marginal revenues (that is equivalent to assuming unconstrained profit maximisation) is made, marginal productivity of the i-th input becomes $(\Delta Y_t - \sum_{j \neq i}^{N} p_{tj}/p_{y} \Delta x_{tj})/\Delta x_{ti}$, which clearly resembles the two-factor (K–L) counterpart proposed by Squire and Van der Tak (1975, p. 111). As can be seen from the context, the measure developed here is superior as it is applicable to the cases with more than two inputs and the more restrictive assumption of unconstrained profit maximisation is not needed.

Given (6), the production elasticity in period t with respect to the i-th input, ξ_{ti}, can be estimated by

$$E_{ti} = \frac{\Delta Y_t p_{ti}}{\sum_{j=1}^{N} p_{tj} \Delta x_{tj}} \frac{x_{ti}}{Y_t}, \tag{7}$$

and the scale elasticity λ between production periods t and $t+1$ can be estimated by

$$\begin{aligned} \epsilon_{t,t+1} &= \sum_{i=1}^{N} E_{ti} \\ &= \frac{\Delta Y_t}{Y_t} \frac{\sum_{i=1}^{N} p_{ti} x_{ti}}{\sum_{i=1}^{N} p_{ti} \Delta x_{ti}}. \end{aligned} \tag{8}$$

3 Validating the Scale Measure

In this section, how precise the proposed estimate of scale elasticity is will be examined, assuming a homogeneous production function and cost minimisation. The reason for focusing on scale elasticity is attributable to the importance of its measurement (Solow 1961, p. 67). Also, by so doing, it would help validate the other measures developed in the last section as the formula for scale elasticity was 'built' on those measures.

Denoting the degree of homogenity of f by λ and let $X_{t+1} = KX_t$, then $Y_t = f(X_t)$ and $Y_{t+1} = f(KX_t) = K^{\lambda}Y_t$ can be obtained. The true scale parameter λ, according to (8), can be estimated by

$$\begin{aligned} \epsilon_{t,t+1} &= \frac{\Delta Y_t}{Y_t} \frac{\sum_{i=1}^{N} p_{ti}/p_{tj} x_{ti}}{\sum_{i=1}^{N} p_{ti}/p_{tj} \Delta x_{ti}} \\ &= \frac{Y_{t+1} - Y_t}{Y_t} \frac{\sum_{i=1}^{N} x_{ti}(\partial f/\partial x_{ti})/(\partial f/\partial x_{tj})}{\sum_{i=1}^{N}(x_{t+1i} - x_{ti})(\partial f/\partial x_{ti})/(\partial f/\partial x_{tj})}. \end{aligned}$$

Taking advantage of Euler's theorem, it is easy to show that

$$\begin{aligned} \epsilon_{t,t+1} &= \frac{K^{\lambda}Y_t - Y_t}{Y_t} \frac{\lambda Y_t}{K\lambda Y_t - \lambda Y_t} \\ &= \frac{K^{\lambda} - 1}{K - 1} \end{aligned} \tag{9}$$

$$\begin{aligned} &= \lambda + \frac{(\lambda - 1)\lambda(K - 1)}{2} \\ &\quad + \frac{(\lambda - 2)(\lambda - 1)\lambda(K - 1)^2}{6} + 0^3[K - 1], \end{aligned} \tag{10}$$

where the last equality is obtained by applying Taylor series expansion at $K = 1$. Clearly, (i) if $\lambda = 1$, $\epsilon = \lambda = 1$; (ii) as $\lambda \to 1$, $\epsilon \to \lambda$; and (iii) as $K \to 1$ (i.e., $X_{t+1} \to X_t$), $\epsilon \to \lambda$.

To show numerically how well the scale estimate approximates its true value, equation (9) is used to calculate ϵs for given λs and Ks. It is unlikely that a firm would change its inputs by more than 35% between consecutive production periods under normal circumstances, so K is confined to be between 0.65 and 1.35. The results are presented in Table 1. It can be seen that for each λ, the average estimation error is almost nil (the maximum being 0.005 for $\lambda = 1.5$).

[Table 1 here]

Plotting the error or estimation bias $\lambda - \epsilon$ and ϵ against λ, Figure 1 shows that (i) in the cases of contraction with decreasing returns to scale or expansion with increasing returns to scale, the scale elasticity will be slightly over-estimated; the countrary is also true. Viewing the prevalence of business cycle, the mean of the estimates of scale elasticity is expected to be very close, if not equal, to the true elasticity; (ii) as long as the true scale elasticity is not greater than 1.3, the bias in the estimates is minimal; and (iii) as long as the changes in inputs is not greater than 15% (contraction or expansion), the bias in the estimates is of little significance. It is thus concluded that the estimates proposed in this paper can provide good approximation to the true production and scale elasticities.

[Figure 1 here]

4 Applications: Production and Scale Elasticities in the US and Japanese Manufacturing

In two important articles, Norsworthy and Malmquist (1983) and Chavas and Cox (1990) estimated productivity changes in the US and Japanese manufacturing. Like many other studies on technical progress, their analyses were based on the assumption of constant returns to scale and cost minimisation. Clearly, unless constant returns to scale did prevail, their estimates would be biased. To test the validity of this commonly-used assumption, data from Norsworthy and Malmquist (1983) are used to calculate scale and production elasticities according to equations (7) and (8). Four inputs are considered. They are capital (K), labour (L), energy (E) and materials (M).

The results are presented in Tables 2 and 3. Production elasticities with respect to these inputs are presented in columns 2-5 and the far-right column shows the scale elasticity. According to the scale estimates, the manufacturing sector experienced increasing returns to scale in 16 out of 19 years for the US and in 7 out of 13 years for Japan; meanwhile decreasing returns to scale prevailed in the remaining years. In other words, disregarding statistical significance, constant returns to scale did not appear at all.

[Tables 2 & 3 here]

However, the sample mean of the scale estimates for Japan is 1.013 and the sample standard error is 0.271. Assuming that the scale parameter follows a normal distribution, it may be concluded that returns to scale are constant for Japan at any conventional significance level. For the US, the sample mean is 1.297 and the sample standard error is 0.506. It is thus easy to conclude that at 1 per cent significance level, the US manufacturing sector displayed increasing returns to scale during the period 1958-76. Given these results, technical progress in the US manufacturing sector must have been overestimated by those who assumed away scale effects.

To comment on the production elasticities briefly, the results indicate that outputs are most responsive to material inputs and least responsive to energy input in both the US and Japan. This explains why in recent years so much importance has been placed on material inputs (see Jorgenson et al. 1987). While in Japan output is more sensitive to changes in capital rather than labour input, the opposite is true for the US. This justifies the finding that Japan experienced much faster growth in capital input than the US did (Norsworthy and Malmquist 1983).

5 Summary

Non-parametric estimates of production elasticity and returns to scale are derived in this paper. The precision of the estimates is examined under a homogeneous function. When applied to the US and Japanese manufacturing data, it is found that increasing returns to scale prevailed in the US and constant returns to scale existed in Japan.

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Figure 1: Scale Elasticity and Its Estimate

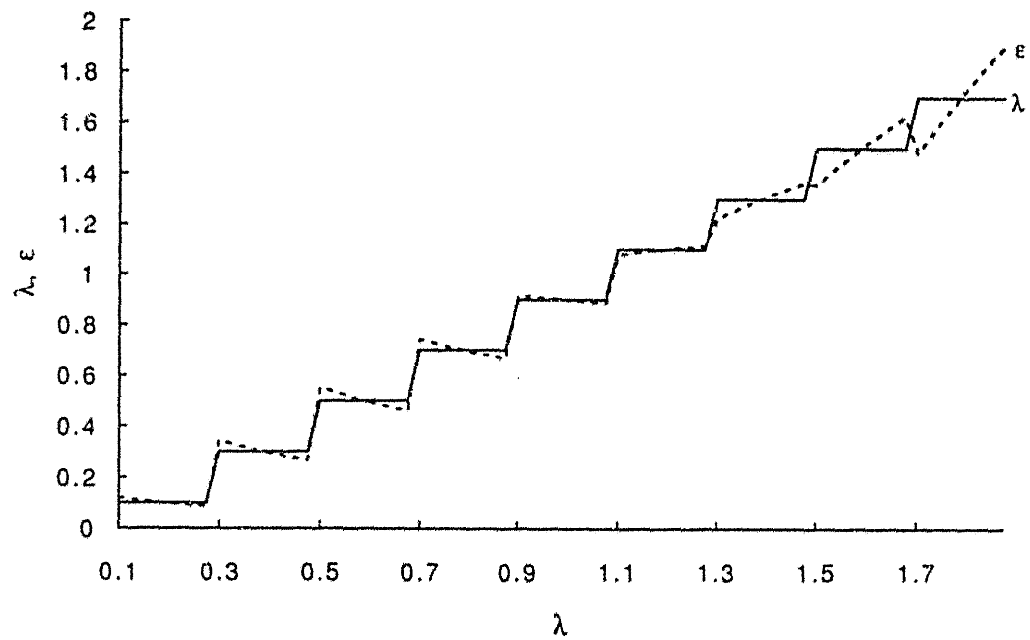

Table 1 Scale Elasticity and Its Estimate for Homogeneous Functions

λ	K	ϵ	$\lambda-\epsilon$	λ	K	ϵ	$\lambda-\epsilon$
0.1	0.65	0.12	-0.02	0.9	1.05	0.90	0.00
0.1	0.75	0.11	-0.01	0.9	1.15	0.89	0.01
0.1	0.85	0.11	-0.01	0.9	1.25	0.89	0.01
0.1	0.95	0.10	0.00	0.9	1.35	0.89	0.01
0.1	1.05	0.10	0.00	1.1	0.65	1.08	0.02
0.1	1.15	0.09	0.01	1.1	0.75	1.09	0.01
0.1	1.25	0.09	0.01	1.1	0.85	1.09	0.01
0.1	1.35	0.09	0.01	1.1	0.95	1.10	0.00
0.3	0.65	0.35	-0.05	1.1	1.05	1.10	0.00
0.3	0.75	0.33	-0.03	1.1	1.15	1.11	-0.01
0.3	0.85	0.32	-0.02	1.1	1.25	1.11	-0.01
0.3	0.95	0.31	-0.01	1.1	1.35	1.12	-0.02
0.3	1.05	0.29	0.01	1.3	0.65	1.23	0.07
0.3	1.15	0.29	0.01	1.3	0.75	1.25	0.05
0.3	1.25	0.28	0.02	1.3	0.85	1.27	0.03
0.3	1.35	0.27	0.03	1.3	0.95	1.29	0.01
0.5	0.65	0.55	-0.05	1.3	1.05	1.31	-0.01
0.5	0.75	0.54	-0.04	1.3	1.15	1.33	-0.03
0.5	0.85	0.52	-0.02	1.3	1.25	1.35	-0.05
0.5	0.95	0.51	-0.01	1.3	1.35	1.36	-0.06
0.5	1.05	0.49	0.01	1.5	0.65	1.36	0.14
0.5	1.15	0.48	0.02	1.5	0.75	1.40	0.10
0.[illegible]	1.25	0.47	0.0[illegible]	1.5	0.85	1.44	0.06
0.5	1.35	0.46	0.04	1.5	0.95	1.48	0.02
0.7	0.65	0.74	-0.04	1.5	1.05	1.52	-0.02
0.7	0.75	0.73	-0.03	1.5	1.15	1.55	-0.05
0.7	0.85	0.72	-0.02	1.5	1.25	1.59	-0.09
0.7	0.95	0.71	-0.01	1.5	1.35	1.62	-0.12
0.7	1.05	0.69	0.01	1.7	0.65	1.48	0.22
0.7	1.15	0.69	0.01	1.7	0.75	1.55	0.15
0.7	1.25	0.68	0.02	1.7	0.85	1.61	0.09
0.7	1.35	0.67	0.03	1.7	0.95	1.67	0.03
0.9	0.65	0.92	-0.02	1.7	1.05	1.73	-0.03
0.9	0.75	0.91	-0.01	1.7	1.15	1.79	-0.09
0.9	0.85	0.91	-0.01	1.7	1.25	1.85	-0.15
0.9	0.95	0.90	0.00	1.7	1.35	1.90	-0.20

Table 2 Production and Scale Elasticities in the US Manufacturing

	Production Elasticity w.r.t.				Scale
Year	Capital	Labour	Energy	Material	Elasticity
1958	0.134	0.362	0.021	0.818	1.334
1959	0.090	0.232	0.013	0.515	0.850
1961	0.148	0.387	0.022	0.853	1.410
1962	0.204	0.509	0.029	1.111	1.852
1963	0.172	0.405	0.023	0.893	1.493
1964	0.141	0.323	0.018	0.709	1.191
1965	0.135	0.292	0.015	0.649	1.091
1966	0.082	0.187	0.009	0.412	0.690
1967	0.139	0.345	0.017	0.755	1.256
1968	0.115	0.288	0.014	0.620	1.037
1969	0.120	0.339	0.016	0.727	1.201
1970	0.242	0.790	0.041	1.664	2.735
1971	0.128	0.377	0.021	0.821	1.347
1972	0.112	0.320	0.018	0.721	1.171
1973	0.027	0.083	0.005	0.196	0.312
1974	0.133	0.520	0.040	1.295	1.988
1975	0.092	0.312	0.028	0.838	1.270
1976	0.094	0.295	0.028	0.788	1.206

Table 3 Production and Scale Elasticities in Japanese Manufacturing

Year	Production Elasticity w.r.t.				Scale
	Capital	Labour	Energy	Material	Elasticity
1965	0.[illegible]74	0.123	0.036	0.726	1.158
1966	0.293	0.128	0.035	0.759	1.215
1967	0.315	0.124	0.033	0.756	1.228
1968	0.288	0.115	0.029	0.687	1.118
1969	0.264	0.104	0.024	0.605	0.997
1970	0.215	0.085	0.019	0.484	0.802
1971	0.349	0.147	0.032	0.785	1.314
1972	0.179	0.082	0.017	0.410	0.687
1973	0.234	0.108	0.022	0.556	0.919
1974	0.108	0.050	0.015	0.272	0.445
1975	0.300	0.174	0.058	0.879	1.411
1976	0.177	0.106	0.032	0.521	0.836
1977	0.226	0.131	0.040	0.645	1.042