



**AgEcon** SEARCH  
RESEARCH IN AGRICULTURAL & APPLIED ECONOMICS

*The World's Largest Open Access Agricultural & Applied Economics Digital Library*

**This document is discoverable and free to researchers across the globe due to the work of AgEcon Search.**

**Help ensure our sustainability.**

Give to AgEcon Search

AgEcon Search  
<http://ageconsearch.umn.edu>  
[aesearch@umn.edu](mailto:aesearch@umn.edu)

*Papers downloaded from **AgEcon Search** may be used for non-commercial purposes and personal study only. No other use, including posting to another Internet site, is permitted without permission from the copyright owner (not AgEcon Search), or as allowed under the provisions of Fair Use, U.S. Copyright Act, Title 17 U.S.C.*

# **Common Knowledge: Removing Uncertainty in Risk Preference Assessments<sup>1</sup>**

Thomas W. Sproul, Assistant Professor

Department of Environmental & Natural Resource Economics

University of Rhode Island

This Draft: July 9, 2012

PRELIMINARY DRAFT – PLEASE DO NOT CITE

Submitted to the Selected Presentations of the 2012 Annual Meeting of the  
Agricultural & Applied Economics Association in Seattle, WA, August 12-14, 2012.

## **Introduction**

An emerging line of research suggests the possibility of replacing crop revenue insurance for US farmers with markets for options contracts tied to the realization of farm revenues. The derivatives markets can be shown to address concerns of basis risk, while removing participation subsidies to insurance carriers, and they are robust to keeping (or not) the large scale transfers via subsidy which are currently in place. In research I presented at the SCC-76 risk group annual meeting in March,<sup>2</sup> it is shown that a derivatives market leads to unique no-arbitrage prices in a general equilibrium setting. A key contribution of that research is introducing a new

---

<sup>1</sup> Working title at time of abstract submission. Likely re-title: Reference dependent preferences in financial markets.

<sup>2</sup> The Annual Meeting of the SCC-76 "Economics and Management of Risks in Agriculture and Natural Resources" Group, March 15-17, 2012 in Pensacola, Florida.

technique for estimating the equilibrium options prices arising from a *known* probability distribution of revenues, the underlying asset to the options.

The next logical step is to invert this process and apply it to financial markets data, thereby deriving the ‘agreed’ underlying probability distribution of returns from observed options prices. However, the ability to invert this process requires an intimate understanding of the risk preferences of market participants and the distribution of those preferences within the population. In the aforementioned study (and in many similar ones), the existence of a financial assets general equilibrium requires an assumption of a concave utility function over wealth. The assumption is problematic because it forces risk aversion over all gambles, contradicting the research in areas of prospect theory and loss aversion, and it rules out the presence of speculators or professional traders known to exist in, and influence, the marketplace (Allen, 2001), who may be risk-loving or have irregular preferences for skewed returns. Thus, while we would like to back out the behavior of market participants in search of some common ‘agreement’ on the underlying probability distribution of returns, the assumptions that might allow us to do so are unpalatable and would likely lead to inconsistent or incorrect results.

In the present article, I present a simple model of a two-period game, which can be solved for a general equilibrium under certain conditions and assumptions about risk preferences. The model compares and contrasts the results obtained in a financial exchange economy under an assumption of expected utility maximizing agents, with concave utilities over wealth, against the results arising from a model of loss-averse agents with non-linear probability weights, as described in Kahneman

and Tversky (1992), and Shumway (1997) among others. The necessary conditions do provide some strong insight into how we might expect options prices to appear when investors are loss-averse, but we show in a simple case of two agents and two states of nature that no equilibrium can exist except a degenerate one.

The next section provides more background on the problem at hand, and the unsuitable aspects of existing approaches. This is followed by a brief review of how loss-averse utilities are modeled. Following that, we introduce a simple exchange economy model, and characterize the equilibrium, first, with a standard expected utility approach, and then, with an approach assuming loss aversion. Given observations of departures from expected utility in actual options markets, we develop the loss-aversion model to gain better explanatory power for observed market phenomena. However, the non-existence of non-degenerate equilibrium ultimately leads us to conclude that perhaps an ex-equilibrium analysis is necessary for the proper evaluation of risk preferences in financial markets.

In the concluding section, I discuss further steps in this line of research. In particular, the implementation of numerical solvers for larger dimensional versions of the basic general equilibrium model, or of agent-based modeling to simulate live trading, to at least produce an intuition about how market prices equilibrate (and/or evolve) when all agents actually do behave according to the loss-averse preferences described herein. Finally, I discuss the design and execution of a pilot financial markets experiment, which removes market imperfections and information asymmetry as obstacles to evaluating risk preferences.

## Why study financial markets to learn about risk preferences?

We focus on derivatives markets, particularly options, because the wide range of strike prices potentially allows for evaluating risk preferences over much of the support of the underlying utility function. The assumption of no arbitrage in efficient markets also forces options prices to reveal an underlying ‘risk-adjusted’ probability distribution, since no arbitrage implies that equilibrium prices must be actuarially fair according to some probability distribution *equivalent* to the underlying physical/actual distribution.<sup>3</sup>

Using an Arrow-Debreu exchange economy model (a simple general equilibrium framework), DeMange & Laroque (2006) showed a key relationship between the aggregate level of wealth and the equilibrium no arbitrage price. This relationship is summed up by the equation (a first-order condition):

$$u' = \lambda \cdot \mu_n$$

where  $u'$  is the first derivative of the utility function,  $\lambda$  is the shadow price of the agent’s budget constraint (arising from asset endowments) in the utility-max Lagrangian, and  $\mu_n$  is the ratio of the equilibrium price,  $q_n$ , to the actuarially fair price (probability),  $p_n$ , of state  $n$ . Given increasing, concave utilities, unique equilibrium prices are known to exist, and  $\lambda$  is constant for each agent, implying a monotone increasing correspondence between the equilibrium state price,  $q_n$ , and  $u'$ . This implies that state prices are higher when  $u'$  is higher, corresponding to lower wealth – according to the assumption of concave, increasing utility.

---

<sup>3</sup> *Equivalent* is defined as a probability distribution having identical regions of zero and non-zero support over its domain.

Finally, since the state prices must sum to one under no-arbitrage, we find that low-wealth states will have higher state prices than actuarially fair (i.e., as indicated by the underlying probability distribution) and high-wealth states will have correspondingly lower state prices than fair. This is borne out in the options prices, as discussed below. However, the implications of this analysis are not robust to misspecification of the utility function. If agents' utilities include regions of convexity (risk-loving), then there could be non-unique or non-existent equilibrium prices, and any equilibrium prices that do exist could behave pathologically. Namely, the relationship between  $q_n$  and  $u'$  might imply negative risk premiums where high-wealth states of nature have prices above actuarially fair. Since there is some evidence of exactly this phenomenon in options markets, we will explore this point in more detail below.

### **Why can't we just use trading data from options markets?**

First, apparently pathological behavior. Standard expected utility models, those which assume a summation over probabilities and which assume an underlying utility function which is concave and increasing in income, clearly do not tell the whole story. In a general equilibrium setting, it can be shown that preferences according to these models generate an intuitive set of prices, whereby the party taking on risk must be paid a premium by the party hedging it away. However, in options markets, the absence of arbitrage coincides with these preferences to produce prices that are *systematically* different from actuarially fair. In the finance literature, the alternative probability distribution which characterizes these prices

is known as the 'risk-neutral' distribution, though perhaps 'risk-adjusted' is a less confusing moniker.

It is not only that the equilibrium prices may be different from their expected value, but that the equilibrium price of the underlying asset, as well as the prices of *all* call options, will be below actuarially fair, while the prices of *all* put options will be above actuarially fair. These are perfectly reasonable conclusions from the expected utility model, but they simply do not match what we observe in financial markets.

Alternately referred to as the volatility skew or volatility smile, it has been well documented since the 1987 crash that put options trade at a risk premium – a risk premium which is increasing as strike prices move further out of the money. To be clear, this risk premium is increasing as a share of the actuarially fair price; it is not increasing in absolute level. The name 'smile' seems to arise intuitively from the use of the Black-Scholes options pricing model, which assumes the underlying asset's price process to be a Geometric Brownian Motion (GBM) and reduces each option price to a function of the volatility (standard deviation) of that process. This model therefore 'explains' the departures from log-normality in the left-hand tail of the distribution as another log-normal with higher implied volatility.

This skewed distribution of returns relative to log-normal is one of the standard justifications for a model of risk averse investors (among other arguments, like the CAPM). However, this phenomenon is substantially more pronounced for 'downside-risk', where investors fear a loss of wealth relative to the reference point of the current price of the underlying asset. Beyond the evaluation problems

inherent in reconciling market prices to the 'actual' probability distribution of returns, there is a lack of agreement about whether market prices are above or below actuarially fair, and by how much, when evaluating options near the money, and out of the money (OTM) calls (e.g., Jackwerth, 2000; Coval and Shumway, 2001). As a result, it may be questioned whether everywhere concave utility functions are really appropriate for modeling investor sentiment towards market risk.

In addition to the challenge of evaluating risk preferences outside traditional models, there is the issue of evaluating them through the filter of financial markets. With regards to data quality, one would hope to obtain (at a minimum) market prices (and trade quantities) of options over a set of strike prices, though in an ideal situation, data could be had on every bid and trade of every market participant. However, even in the ideal situation, there remain issues of market power, transaction costs, access to markets, and information asymmetry, which can distort observed prices and quantities away from theoretically predicted equilibrium values. Each of these real world challenges thus acts as yet another barrier to evaluating risk preferences with data from actual financial markets.

Finally, the largest obscuring force is probably aggregation (Campbell, 2000). Without the ability to observe and track individual market participants, all that remains are faceless trades of a representative consumer. While some smoothing techniques may help to identify 'equilibrium' prices from anonymous market transactions, the practitioner is left to piece together preferences with very little help. Constantinides (1982) and others since have helped identify to what extent a representative consumer approach can help to tease out individual preferences,



especially when all agents have convex preferences (i.e., concave utility functions) and when they deal adequately with evaluating probabilities. The observational data we have described above are not consistent with these preferences, so further work will be needed. Nonetheless, the representative consumer model coupled with our observations about price distortions can rule out the possibility that *every* agent in financial markets is an expected utility maximizer with a concave utility function, opening up the possibility to explore other approaches regarding decision-making under risk.

The next section develops an economic and financial framework of farm revenue securities and options, and implement a model of a financial exchange economy to establish options prices and risk premiums vs. actuarially fair. We develop a classical exchange-economy equilibrium framework using financial economics and no-arbitrage pricing theory (particularly, Varian 1987, Karatzas et al. 1990, DeMange and Laroque 2006, and Delbaen and Schachermayer 2006), and use it to parameterize equilibrium prices and demands for financial assets.

### **A Financial Market Model of Options Trading**

***Pricing by No-Arbitrage.*** Consider an asset with an uncertain payoff, for example, ownership of a share of stock (though for simplicity, it pays no dividends). At time 0, the asset has a market price,  $S_0$ . At time 1, the value of the asset is given by the realization of a random variable,  $S_1$ , representing the prevailing market price of the stock. The ex ante market price  $S_0$  need not be equal to the expected value of the

stock price,  $E[S_1]$ , and because of risk aversion in the marketplace (under the traditional expected utility model) we would expect it to be lower. Assume that the random future stock price variable,  $S_1$ , takes on a finite number of non-negative values, which represent dollar denominations. Formally,  $S_1$  is mapped to a finite probability space,  $(\Omega, \mathcal{F}, \mathbf{P})$ , where  $\Omega$  is the set of all possible stock price outcomes,  $\mathcal{F} = 2^\Omega$  is a filtration (in this case, the set of all its subsets), and  $\mathbf{P}$  is a probability measure. Without loss, we assume that all outcomes  $\omega \in \Omega$  have strictly positive probability.

Now, consider an agent purchasing a European-style put option with strike price,  $K$ . Once  $S_1$  is realized and the stock price outcome is known, the agent is paid only if the realized price is below the strike ( $S_1 < K$ ). If this occurs, then the writer pays the difference,  $K - S_1$ , to the agent, in effect restoring him to a guaranteed level of wealth. We denote the value of this payment as  $(K - S_1)^+ = \max\{0, K - S_1\}$ .<sup>4</sup>

In order to evaluate how a simple market for these options might work, we make a number of assumptions. First, we assume that markets are complete and frictionless. That is, any agent can trade options at any strike price, the number of contracts exchanged is real-valued and perfectly divisible, and there are no transaction costs, information asymmetries, or distortions caused by market power. Second, we assume that there are no opportunities for arbitrage. This is the classic

---

<sup>4</sup> We have chosen our notation deliberately to match the options standard notation in the finance literature. Any options referred to in this article are European-style options; there is no possibility of early exercise. Please see Part 1: Basic Option Theory in Wilmott et al. (1995) for a thorough review.

assumption that there can be no risk-free profits, which implies that any financial portfolios with identical payoffs in all states of nature must have the same cost. A common justification is that agents exploiting arbitrage opportunities will bid prices up until the opportunities disappear, so that no arbitrage opportunities can exist *in equilibrium*. For clarity of exposition, we also set interest and discount rates equal to zero.

Our assumptions of no arbitrage opportunities and market completeness are quite powerful: complete markets assure a complete set of Arrow securities, denoted  $1_n$ , paying exactly one dollar in state of nature  $n$  and zero otherwise (DeMange and Laroque, 2006, among many examples). Since replicated payoffs always have the same price, all possible contingent claims are priced at time 0 exactly equal to the sum of the prices of the Arrow securities needed to replicate them. Following the literature, we dub the prices of the Arrow securities as state prices,  $q_n = q_n(1_n)$ . By no arbitrage, all the state prices must be strictly positive ( $q_n > 0$ ) and they must sum exactly to one because buying one of each will pay one dollar with certainty (the price of one dollar in the future period, under our assumption of a zero interest rate).

Thus, the state prices  $\{q_n\}_{n=1,\dots,N}$  have the properties of a probability mass function, which we will refer to as the risk-adjusted probability measure,  $\mathbf{Q}$ . The measure  $\mathbf{Q}$  is known to be unique, given complete markets with prices not subject to arbitrage (Delbaen and Schachermayer, 2006). Accordingly, the price of any

contingent claim is equal to the weighted sum of state prices needed to duplicate its payoffs, which is simply the actuarially fair price under  $\mathbf{Q}$ . Returning to our model, this means that the ex ante price of the rights to crop revenues must be equal to its fair price under this alternative probability measure, so  $S_0 = E_{\mathbf{Q}}[S_1]$ . However, even though the measure  $\mathbf{Q}$  is unique as a function of the no-arbitrage prices, the prices themselves are not guaranteed to be unique (Varian, 1987): for this we must turn to a model of risk preferences in the marketplace.

In what follows, we evaluate equilibrium in the context of the two-period game outlined above. The efficient markets hypothesis tells us that markets instantaneously equilibrate, so our two-period model is simply the evaluation of markets at a fixed point in time before new information arrives. Not using continuous-time trading and stochastic processes offers two distinct advantages for our purposes here: 1) we are able to take a completely non-parametric approach with respect to the asset returns (don't need to assume normality/log-normality), and 2) as a result, we can directly estimate risk premiums in options prices across all strikes, without resorting to approximations derived from the moments of the returns distribution.

***A Simple Model of a Financial Exchange Economy.*** Consider an economy in which the only assets are cash (which is riskless, but pays no interest), the underlying security, and the complete set of options contracts, where the probabilities of all states of nature are known. In what follows, it will become clear that even if shares of stock cannot be bought and sold (e.g., only options are available), complete,

arbitrage-free options markets are sufficient to force an equilibrium asset price at time 0.<sup>5</sup>

No arbitrage and complete markets allow us to consider everything denominated in terms of the Arrow securities,  $1_n$ . The economic agents,  $J = 1, \dots, J$ , have endowment vectors  $y_j = \{y_{nj}\}_{n=1, \dots, N}$  over the states of nature,  $n$ . For those agents not holding any shares in farm revenues, we model them as holding only cash – equivalent to holding equal amounts of Arrow securities across all states. Agents who hold shares of stock, on the other hand, have larger amounts of endowment in ‘higher’ states (where the period 1 stock price realization is higher), and smaller amounts in ‘lower’ ones. Given an endowment, the agents can sell their endowments (or even take a short position for securities they do not own) to purchase other securities, subject to a budget constraint defined by the initial endowment.

***The Expected Utility Model.*** We start by introducing the results of the basic expected utility model, before moving on to the more complex results with loss aversion. To simplify the model, there is no utility at time 0, and every agent strives to maximize expected utility at time 1, over an increasing, strictly concave utility function of wealth. Market participants maximize expected utility according to a

---

<sup>5</sup> The no-arbitrage assumption implies a property of derivatives markets known as put-call parity, which maintains the relationship  $C + K = S + P$  for all strike prices,  $K$ . This formula allows for constructing “synthetic” ownership of the underlying asset, without actually being able to purchase it. Allowing the asset to be bought and sold preserves the original nature of options, which confer the rights to buy or sell the asset at a specific price. Certainly, options contracts can be (and often are) cash-settled, so allowing trading the assets themselves is a stylistic choice for our modeling approach. In practice, there may also be the social objective of market liquidity, which would benefit from allowing trading in the underlying asset.

budget constraint given by their starting endowment:

$$\max_{\lambda_j, \{x_{nj}\}_{n=1, \dots, N}} E[u] = \sum_n p_n \cdot u(x_{nj}) + \frac{1}{\lambda_j} \sum_n q_n \cdot (y_{nj} - x_{nj})$$

where  $p_n$  is the known state probability,  $q_n$  is the state price, and  $x_{nj}$  is the individual's chosen consumption of (demand for) the Arrow security for state  $n$  (which may be a different amount than his endowment). The first order condition for each  $x_{nj}$  gives:

$$u'(x_{nj}) = \frac{\mu_n}{\lambda_j}$$

where we have generalized the state prices as  $q_n = p_n \cdot \mu_n$ , and where  $\mu_n$  is a positive weight translating the physical probability under the measure  $\mathbf{P}$  into the risk-adjusted probability under the measure  $\mathbf{Q}$  (that is,  $\mu_n$  is the inverse of the Radon-Nikodym derivative,  $d\mathbf{P}/d\mathbf{Q}$ ). Thus, we obtain the formula:

$$x_{nj} = u'^{-1} \left( \frac{\mu_n}{\lambda_j} \right)$$

Strictly concave utilities imply that the first order conditions are necessary and sufficient for existence of a unique equilibrium, provided the utility functions are sufficiently regular over their domains. These results are standard and well established in microeconomic theory; all we have done is generate an exchange economy of Arrow securities, with increasing, concave utility functions and heterogeneous endowments. To make progress in solving for equilibrium prices, the

utility function must be known. We assume each agent has a constant absolute risk aversion (CARA), exponential utility function, with risk aversion coefficient,  $a_j$  :

$$u_j(x) = \frac{-\exp(-a_j \cdot x)}{a_j}$$

Substituting in this utility function, the first order condition for each  $x_{nj}$  thus reduces to:

$$e^{-a_j x_{nj}} = \frac{\mu_n}{\lambda_j} \quad \text{or} \quad x_{nj} = \frac{\ln \lambda_j - \ln \mu_n}{a_j}.$$

So, even though we need to solve for  $N \times J$  equilibrium demands, these are each composed of only the equilibrium weights,  $\mu_n$ , and each agent's shadow price of wealth (the Lagrange multiplier),  $1/\lambda_j$ . To solve for equilibrium demands, we must solve simultaneously only for the  $N$  state prices and the  $J$  shadow prices ( $N + J$  unknowns, instead of  $N \times J$ ). The constraints are given by:

$$(1) \sum_n q_n (\omega_{nj} - x_{nj}) = 0 \quad \forall j$$

$$(2) \sum_j (\omega_{nj} - x_{nj}) = 0 \quad \forall n$$

$$(3) \sum_n q_n = 1$$

The first set of constraints assures that each agent's budget constraint is satisfied with equality, which must hold in equilibrium if agents are utility maximizers. The second constraint requires that the financial instruments are in zero net-supply, that is, that total consumption is equal to the total endowment in each state of

nature. The final constraint recognizes that prices are only unique up to a positive multiplicative factor, and forces the unique, no-arbitrage, equilibrium prices.

The equilibrium state prices,  $q_n$ , imply unique option prices according to the risk-adjusted probability measure,  $\mathbf{Q}$ , which can then be contrasted against actuarially fair prices under the physical probability measure,  $\mathbf{P}$ . Using summation by parts, the prices of options can be explicitly calculated:

$$\begin{aligned} P_K &= E_Q[\max\{0, K - S\}] \\ &= \sum_{n < K} q_n \cdot (K - \omega_n) \\ &= \sum \sum_{n < K} q_n \Delta \omega_n \end{aligned}$$

where  $\omega_n$  denotes the realization of stock price in state  $n$ , and the differential term,  $\Delta \omega_n = \omega_{n+1} - \omega_n > 0$ , is always positive because the  $\omega_n$ 's are arranged to be increasing in  $n$ . The market/equilibrium (risk-adjusted) price of the options is given by the double sum over  $q_n$ , whereas the actuarially fair price,  $\tilde{P}_K$ , is simply the double sum over  $p_n$ .

Letting  $\tilde{P}_K$  denote the put option priced at actuarially fair, the risk premium is given by:  $(P_K - \tilde{P}_K)/\tilde{P}_K$ . In a working paper, Sproul and Rausser (2012), showed that the preferences described generate positive risk premiums which are decreasing in the strike price (as a percentage, as defined), though the absolute amount paid above actuarially fair to offload risk,  $P_K - \tilde{P}_K$ , is increasing in the strike price. To the contrary, the concave utility functions described induce negative risk premiums on call options, which are price reductions below actuarially fair. The call



options trade below actuarially fair because an agent not holding the underlying stock must be compensated for taking on risk when buying a call, and agents who do hold the underlying are willing to pay to offload their risk when writing a call. The call options have the opposite trending characteristic, where the risk pricing distortion (percentage) increases in the strike price, while the absolute amount paid to offload risk decreases as the strike price rises. As we discussed above, put-call parity allows for synthetic construction of share ownership by buying a call and selling a put option, so the synthetic share price in Period 0 (and by no-arbitrage, the actual share price) is subsidized by the sum - a constant amount over all strike prices - of the dollar risk discount on the purchase of the call option plus the dollar risk premium (amount, not percentage) on the sale of the put option.

### **A Model of Loss-Averse Value Functions.**

As discussed in the introductory sections, the patterns of risk premiums outlined above do not coincide well with documented, consistent price deviations from actual returns. While it is well known that buying out of the money put options produces strongly negative returns, this is the only price deviation pattern consistent with the von-Neumann Morgenstern (vNM) concave expected utility model. Coval and Shumway (2001) document positive returns for call and put options at intermediate levels of 'money-ness', and Jackwerth (2000) documents consistent negative returns at some levels of out of the money call options.

All of these patterns violate the vNM expected utility model, but all are explained consistently within the standard loss aversion framework of Kahneman

and Tversky (1992). Specifically, we refer to their documented ‘fourfold pattern of risk attitudes’ which appears over and over again in experiments, and seems to show up in options markets as well: the interaction of non-linear probability weighting and loss averse value functions leads to risk-averse behavior for rare/low-probability losses (including extreme ones), and for moderate to high probability gains (usually smaller). Similarly, the loss-averse specification implies risk-seeking behavior for low-probability gains (including large ones), as well as for moderate to high probability losses, including more frequent losses of smaller size. Coval and Shumway (2005) document exactly these patterns in bond-trading market makers at the Chicago Board of Trade (CBOT), who demonstrate characteristic loss-averse responses to fluctuations in intra-day profitability.

Given this background, consider the following model of reference-dependent preferences introduced by Kahneman and Tversky (1992) and generalized (beyond our purposes here) by Koszegi and Rabin (2006). We describe the value function as a power function characterizing the phenomenon of ‘diminishing sensitivity’, exhibiting a kink at the reference point and loss-aversion:

$$v(x) = \begin{cases} (x-r)^a & x > r \\ -\lambda \cdot (r-x)^b & r > x \end{cases}$$

where  $a, b \in (0,1)$ ,  $\lambda > 1$ , and the first derivative is not continuous at the reference point,  $x = r$ . From the psychology and economics literature,  $\lambda = 2.25$  is a standard estimate (population median), as is  $a = b = 0.88$ . We will adopt the convention  $a = b$  in what follows to simplify notation.

Beyond the value function, the other piece in the puzzle of fourfold risk attitudes is the tendency of decision-makers to weight probabilities non-linearly. This weighting is approximated by Kahneman and Tversky (1992) and many others as a monotonic increasing two-part power function, with distinct coefficients,  $\gamma$  and  $\delta$ , applied for weighting gains and losses,  $w^+(p)$  and  $w^-(p)$ , respectively:

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}} \text{ and } w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{1/\delta}}.$$

A common approximation (including in earlier versions – 1979 – of their own theory) is to set  $w^+ = w^-$ , even though  $\gamma$  and  $\delta$  have been estimated at 0.61 and 0.69, respectively. In translating these weights to our previous model with physical probability measure  $\mathbf{P}$ , we will denote  $w_n^+ = w^+(p_n)$  and the same for  $w_n^-$ . The key challenge of modeling  $\gamma$  and  $\delta$  distinctly is that these weights are conditional on whether a gain or loss is occurring relative to the reference point in state  $n$ , and this outcome is endogenous to the financial market trading decisions of the agent. We will explore the implications more carefully in what follows.

Returning to our model of Arrow securities (arising from a complete options market), agents seek to maximize their value function relative to a reference point. That is, they seek to maximize:

$$\max_{\{x_{nj}\}} V_j = \sum_n w_n^+ \cdot v\left((x_{nj} - r)^+\right) + \sum_n w_n^- \cdot v\left((x_{nj} - r)^-\right)$$

where the “+” and “-” superscripts inside the value function denotes that the  $x_{nj} - r$  term is non-zero only if it is positive or negative, respectively, and the “+” and “-”

superscripts on the weighting terms,  $w_n$ , indicate the possibly different gain-loss probability weights.

A challenge immediately arises of how to evaluate the reference point. For example, given perfectly liquid markets, any portfolio could be liquidated at cash value and used to purchase equal amounts of Arrow securities in all states – yielding a fixed income with certainty in Period 1. Does it make sense to use this cash value as a reference point? The simple answer is no. Using the cash value as a reference point is only appropriate if the agent starts out with an ‘all-cash’ constant portfolio of Arrow securities across states. Extending beyond our model to dynamic time considerations, we recognize that for an agent to hold a non-constant portfolio he must prefer it to cash in some sense. However, in the simple two-period game described here, we focus on trading choices given an initial assigned endowment of Arrow securities, which may or may not be constant across states.

Given the status quo preference implied by loss aversion, it certainly does not make sense to benchmark portfolios to their cash value (or any fixed value) because the constant benchmark would get the reference dependence wrong. A simple example here should suffice. Consider an agent switching from a portfolio,  $\mathbf{x}_j$ , to another portfolio with fixed value,  $y$ , and back again. The value of this double-switch is given by  $V(y; \mathbf{x}) + V(\mathbf{x}; y)$  which is shown to be less than zero in the equation below. This non-transitivity of preferences is exactly why any constant benchmark cannot work.

$$\begin{aligned}
V(y; \mathbf{x}) + V(\mathbf{x}; y) &= \sum_{n: x < y} w_n^+ \cdot v\left((y - x_{nj})^+\right) - \sum_{n: x > y} w_n^- \cdot \lambda \cdot v\left((x_{nj} - y)^+\right) \\
&\quad + \sum_{n: x > y} w_n^+ \cdot v\left((x_{nj} - y)^+\right) - \sum_{n: x < y} w_n^- \cdot \lambda \cdot v\left((y - x_{nj})^+\right) \\
&= (1 - \lambda) \cdot \left( \sum_{n: x < y} w_n^+ \cdot v\left((y - x_{nj})^+\right) - \sum_{n: x > y} w_n^- \cdot \lambda \cdot v\left((x_{nj} - y)^+\right) \right) \\
&< 0
\end{aligned}$$

Under this limitation, it remains to properly evaluate the reference point when that reference point is a lottery, i.e., when an agent is assigned a non-constant portfolio in Period 0, so they start out expecting a stochastic outcome. Koszegi and Rabin (2006) assert that the realized outcome in any state is to be compared against all possible reference states according to their distribution, because agents do care about ‘what could have been’ in their reference lottery. Given a stochastic endowment,  $\mathbf{y}$ , their claim amounts to evaluating:

$$V(\mathbf{x}; \mathbf{y}) = \sum_{n: x > \mathbb{E}[\mathbf{y}]} w_n^+ \cdot v\left((x_{nj} - \mathbb{E}[\mathbf{y}])^+\right) - \sum_{n: x < \mathbb{E}[\mathbf{y}]} w_n^- \cdot \lambda \cdot v\left((\mathbb{E}[\mathbf{y}] - x_{nj})^+\right)$$

where  $\mathbb{E}[\mathbf{y}]$  is presumably evaluated according to the perceived distribution of outcomes in portfolio  $\mathbf{y}$ .

However, if the fourfold risk attitudes of Kahneman and Tversky are to be believed, then we are presented with a dilemma of defining the correct probability distribution. Koszegi and Rabin favor using the physical distribution,  $\mathbf{P}$ , which is consistent with their model using linear probability weights, but for our purposes here it seems logically inconsistent to use non-linear probability weights on a prospect while using linear weights on a reference. Furthermore, as currently defined,  $w_n^+$  and  $w_n^-$  vary according to whether the outcome is a gain or a loss, which

begs the problematic question: what is the reference point for the reference point,  $y$ ? The problem here is, since the probability weights are tied up in any prospect's reference point, every *stochastic* reference point must itself be evaluated in terms of a reference point one step further up the chain.

To make progress, we employ simplified probability weights as in Shumway's (1998) working paper, where he verifies goodness-of-fit for loss averse preferences in evaluating investment returns under a simpler version of the non-linear weighting scheme,  $w$ . Specifically, like Shumway, we set  $\gamma = \delta$  so that  $w_n^+ = w_n^- = w_n$  for all states  $n$ . From here, the resulting weights  $w_n$  are consistent across gains and losses, and so can be normalized to sum to one (like a probability measure). In order to avoid further cumbersome notation,  $w_n(p_n)$  will henceforth refer to this alternative probability measure, calculated as:

$$w_n(p_n) = \frac{p_n^\gamma}{(p_n^\gamma + (1 - p_n)^\gamma)^{1/\gamma} \cdot \left( \sum_n \frac{p_n^\gamma}{(p_n^\gamma + (1 - p_n)^\gamma)^{1/\gamma}} \right)}$$

$$= \left( 1 + \sum_{m \neq n} \frac{p_m^\gamma}{(p_n^\gamma + (1 - p_m)^\gamma)^{1/\gamma}} \right)^{-1}$$

where the summation term forces the sum of the weights to one. While the formula looks messy, it simplifies tremendously in some cases. For example, in the case of only two states, it reduces to:

$$w_1 = \frac{p_1^\gamma}{p_1^\gamma + p_2^\gamma}.$$

This formulation allows for applying Koszegi and Rabin's (2006) value function without the ad-infinitum nesting of reference points. Specifically, using the new weighted probability measure,  $\mathbf{W}$ , their formula becomes:

$$V(\mathbf{x}; \mathbf{y}) = \sum_n w_n \cdot v\left((x_{nj} - \mathbb{E}_{\mathbf{W}}[y])^+\right) - \sum_n w_n \cdot \lambda \cdot v\left((\mathbb{E}_{\mathbf{W}}[y] - x_{nj})^+\right)$$

which has the convenient property of boiling down the reference point into one constant value,  $\mathbb{E}_{\mathbf{W}}[y]$ , the expectation under the distribution  $\mathbf{W}$ . Accordingly, agents solve the following problem according to a budget constraint, with the Lagrangian given by:

$$\max_{\{x_{nj}\}} V = \sum_n w_n \cdot v(x_{nj} - \mathbb{E}_{\mathbf{W}}[y]) + \theta \sum_n q_n \cdot (y_{nj} - x_{nj}),$$

where  $\theta$  is the Lagrange multiplier on the constraint. Thus, for each non-zero  $x_{nj}$  in the agent's optimal bundle, we obtain the following necessary first order condition:

$$v'(x_{nj} - \mathbb{E}_{\mathbf{W}}[y]) = \theta \frac{q_n}{w_n}$$

which is remarkably similar in general form to that of the exchange economy equilibrium noted above. The first order condition simplifies to:

$$a(x_{nj} - \mathbb{E}_{\mathbf{W}}[y])^{a-1} = \theta \frac{q_n}{w_n} \quad \text{or} \quad \lambda \cdot a(-x_{nj} + \mathbb{E}_{\mathbf{W}}[y])^{a-1} = \theta \frac{q_n}{w_n}$$

depending on whether the  $x_{nj}$ 's are gains or losses relative to the reference point, respectively. Recall that  $\lambda$  is the loss aversion coefficient, while  $\theta$  is the Lagrange multiplier. Letting  $\mathbf{I}^-$  be an indicator function equal to one if a loss and zero otherwise, all non-zero  $x_{nj}$ 's in the optimal solution are given by:

$$x_{nj} = \left( \lambda^\Gamma \frac{a \cdot w_n}{\theta \cdot q_n} \right)^{\frac{1}{1-a}} + \mathbb{E}_{\mathbf{w}}[y] \cdot (-1)^\Gamma .$$

As noted above in the case of vNM expected utilities, the constraints of budget balance for all agents and options contracts in zero net supply give a system of  $N + J$  non-linear equations in  $N + J$  unknowns. However, in the case of reference dependent preferences, the shape (and kinked nature) of the utility functions do not guarantee the existence of an equilibrium – unique or otherwise. Negative definiteness of each agent's Hessian matrix of second derivatives of the objective function would guarantee the unique equilibrium, but this condition cannot generally be expected to hold. Nonetheless, the first order conditions do tell a story about the relationship between resource scarcity and prices, provided an equilibrium does exist. To see this, define the demand deviate,  $\Delta x_{nj} = x_{nj} - \mathbb{E}_{\mathbf{w}}[y]$ , to be the amount by which each Arrow security deviates from the reference point. Then,

$$\Delta x_{nj} = (-1)^\Gamma \cdot \left( \lambda^\Gamma \frac{a \cdot w_n}{\theta \cdot q_n} \right)^{\frac{1}{1-a}} ,$$

where we recall from the preceding discussion that the Lagrange multiplier,  $\theta$ , is fixed for all agents,  $j$ , so any positive deviate ( $\Delta x_{nj} > 0$ ) will be decreasing in the state price,  $q_n$ , and increasing in the perceived probability,  $w_n$ . The opposite signs will hold for negative deviates (where the amount of Arrow security purchased,  $x_{nj}$ , is below the reference point), so we can express these conditions – in terms of the absolute value of the deviates – as follows:



$$\frac{\partial}{\partial w_n}(|\Delta x_{nj}|) > 0 \text{ and } \frac{\partial}{\partial q_n}(|\Delta x_{nj}|) < 0.$$

Two immediate contrasts follow, between our reference dependent model and the basic expected utility model. While in the standard model demand,  $x_{nj}$  is decreasing in the state price and increasing in the state probability, the preferences outlined here induce different relationships that seem to reflect status-quo bias: *the absolute deviation from the reference point is increasing in perceived state probability and decreasing in perceived state price*. It remains to be seen how or whether price varies systematically with aggregate income, as it does under the expected utility model (it decreases). Given the potential existence problems we discuss above, we turn to a simplified model of equilibrium to gain some basic insights.

### *Equilibrium Under Reference Dependent Preferences*

In this section, we develop a simple 2x2 model (two agents, two states), to help describe the set of equilibria under reference dependent preferences. Specifically, we characterized two types of equilibrium, a ‘degenerate’ equilibrium where the kink in the value function causes all state prices,  $q_n$ , to equal the perceived probabilities,  $w_n$ , and all agents demand exactly their reference point in cash, and a ‘trading’ equilibrium, where agents potentially reach interior maxima of their value functions. In what follows, we rule out the existence of any trading equilibria and then show that the degenerate equilibrium only holds for one ratio of perceived probabilities between the two states. The remainder of this article, up until the discussion section, can thus be considered an exploratory proof of these statements.

Consider the following model of two agents with reference dependent preferences, as described above, in the same two-period game as before. There are two states of nature, with *average* endowments (half of the aggregate) of  $y_L = y$  and  $y_H = \beta y : \beta > 1$ . The first agent (assigned arbitrarily) therefore has an endowment characterized by  $(y + \Delta y_1, \beta y + \Delta y_2)$ , and the second agent's endowment is given by  $(y - \Delta y_1, \beta y - \Delta y_2)$ , where the delta terms indicate deviation from the average endowment. These endowments yield reference points of:

$$\begin{aligned} r_1 &= w_1 \cdot (y + \Delta y_1) + w_2 \cdot (\beta y + \Delta y_2) \\ r_2 &= w_1 \cdot (y - \Delta y_1) + w_2 \cdot (\beta y - \Delta y_2) \end{aligned}$$

where the subscripts on  $r$  denote the agent, but the subscripts on  $y$  and  $w$  denote the state. It will be useful later to note that  $r_2 = 2w_1 y + 2w_2 \beta y - r_1$ .

The presence of the reference point introduces a unique consideration, namely, whether agents can actually afford their reference point. We will show later that if  $\mathbf{q} = \mathbf{w}$  (actuarially fair prices, according to the perceived probabilities) then, depending on endowments, we get the degenerate equilibrium where agents can exactly afford their reference points and they do not trade. However, outside this case, we have to consider whether the agents can collectively, or individually, over-afford or under-afford their collective reference points. Collectively over-affording will simply be defined as  $(\mathbf{q} - \mathbf{w}) \cdot \mathbf{y} > 0$ .

*Proposition: The agents can collectively over-afford their reference points IFF the low state is underpriced relative to its perceived probability ( $q_1 < w_1$ ). Similarly, agents*

*collectively under-afford their reference points IFF the low state is overpriced relative to its perceived probability ( $q_1 > w_1$ ).* The proof follows from simple algebra, upon examining the definition for over-affording.

Beyond the collective ability to afford their reference points, we are also concerned with agents' individual ability to achieve their reference points, a stricter qualification. This qualification matters because inability to afford the reference requires that an agent face a loss in at least one state of nature. Recognizing that  $q_n$  and  $w_n$  are probabilities, agent 1's affordability condition reduces to:

$$\begin{aligned} s_1 &= (q_1 - w_1)(y + \Delta y_1 - \beta y - \Delta y_2) \\ &= (q_1 - w_1)(y_{11} - y_{21}) \geq 0 \end{aligned} ,$$

where the first term is signed according to the relative price of the low-state.

Similarly, agent 2's condition is:

$$s_2 = (q_1 - w_1)(y_{12} - y_{22}) \geq 0 .$$

These two conditions show the simple fact that an agent having more endowment in that state of the nature with a price higher than its perceived probability (aka overpriced) implies over-affordability of the reference.

Thus, when the low-state is underpriced ( $q_1 < w_1$ ), implying the agents can jointly over-afford their references, then the agents can individually over-afford their references only if they have a higher endowment in the high-state ( $y_{2j} > y_{1j}$ ). It is possible for both agents to over-afford in this scenario, since the high-state has a higher (average and therefore, aggregate) endowment, but doing so will require a certain amount of balance between their endowments. On the other hand, when the

low-state is overpriced ( $q_1 > w_1$ ) and agents jointly cannot afford their references, then balanced endowments imply both under-afford individually while unbalanced endowments will allow only the agent with a higher endowment in the low-state ( $y_{1j} > y_{2j}$ ) to over-afford. Again, since  $\mathbf{q}$  and  $\mathbf{w}$  are probability measures, any agent with exactly equal endowments in each state (holding cash) can exactly afford his reference point. The following tables break down the relationship between endowments and prices.

<i>Low-state underpriced: <math>q_1 &lt; w_1</math></i>			
		<b><i>Agent 1</i></b>	
		$y_{11} > y_{21}$	$y_{11} < y_{21}$
<b><i>Agent 2</i></b>	$y_{12} > y_{22}$	<i>N/A</i>	<i>1 over, 2 under</i>
	$y_{12} < y_{22}$	<i>1 under, 2 over</i>	<i>Both over</i>

<i>Low-state overpriced: <math>q_1 &gt; w_1</math></i>			
		<b><i>Agent 1</i></b>	
		$y_{11} > y_{21}$	$y_{11} < y_{21}$
<b><i>Agent 2</i></b>	$y_{12} > y_{22}$	<i>N/A</i>	<i>1 under, 2 over</i>
	$y_{12} < y_{22}$	<i>1 over, 2 under</i>	<i>Both under</i>

With these relationships established, we turn to a more critical question: whether or not agents can afford their reference points, which bundles do they actually choose? Recall the Lagrangian of the value function, which agents maximize with respect to their reference point while constrained by the market price of their

endowments. If we assume an equilibrium outside the degenerate case, then we know neither agent is stuck at exactly his reference point in just one state of nature (i.e.,  $x_{nj} = r_j$  for one state and not the other) because the value function has marginal utilities approaching infinity in the limit as  $x_{nj}$  approaches the reference point. Thus, we can be confident that a non-degenerate equilibrium will have a non-zero Lagrange multiplier for each agent,  $\theta_j > 0$ , because the budget constraints will bind.

As a result, we know that any non-degenerate equilibrium will involve interior solutions for all  $x_{nj}$ , so the first order conditions described above are necessary for an optimum. Given the form of the value functions, we can re-write and combine the first order conditions for both agents in each state:

$$\Delta x_{n1} = f(\mathbf{q}, \mathbf{w}) = k_n \cdot \Delta x_{n2},$$

where  $k_n$  is a constant known to exist (given  $\theta_1, \theta_2$ , and  $\lambda$ ), and  $f$  describes the specific features of the first order condition for state  $n$ . Consider this equation re-written for state 1, for example:

$$\begin{aligned} x_{11} - r_1 &= \left[ m_1 \frac{\theta_1}{\theta_2} \right]^{\frac{1}{a-1}} \cdot (x_{12} - r_2) \\ \rightarrow x_{11} - r_1 &= k_1 \cdot (x_{12} - r_2), \end{aligned}$$

where  $m_1 \in \{1, -\lambda, -\lambda^{-1}\}$  depending on whether either or both agents face a loss or gain in state 1, and  $k_1$  will vary accordingly.

Substituting in the enforced relationship between the reference points (resulting from the division of the starting endowments, and the fact that the two

demands cannot exceed the state 1 endowment in equilibrium (i.e.,  $x_{11} + x_{12} = 2y$ ),

we obtain an explicit solution:

$$\Delta x_{11} = \frac{k_1}{1+k_1} \cdot 2y \cdot (1+\beta) \cdot w_2; \quad \Delta x_{12} = \frac{1}{1+k_1} \cdot 2y \cdot (1+\beta) \cdot w_2.$$

Similarly, for state 2 we obtain:

$$\Delta x_{21} = \frac{k_2}{1+k_2} \cdot 2y \cdot ((1+\beta) \cdot w_2 + \beta - 1); \quad \Delta x_{22} = \frac{1}{1+k_2} \cdot 2y \cdot ((1+\beta) \cdot w_2 + \beta - 1).$$

The signs of all the demand deviates are wholly dependent on the sign of the leading fractional term. The table below gives the correspondence of these terms with  $m_n$ .

Letting  $\vartheta := (\theta_1 / \theta_2)^{\frac{1}{a-1}} > 0$ , the relationships are:

	$m_n = 1$	$m_n = -\lambda$	$m_n = -\lambda^{-1}$
$\frac{k_n}{1+k_n}$	$\frac{\vartheta}{1+\vartheta} > 0$	$\frac{\lambda \cdot \vartheta}{\lambda \cdot \vartheta - 1}$	$\frac{1}{1-\lambda \cdot \vartheta}$
$\frac{1}{1+k_n}$	$\frac{1}{1+\vartheta} > 0$	$\frac{1}{1-\lambda \cdot \vartheta}$	$\frac{\lambda \cdot \vartheta}{\lambda \cdot \vartheta - 1}$

where the second and third columns have alternating signs. In a given state,

$m_n = -\lambda$  means a loss for agent 2 and a gain for agent 1, while  $m_n = -\lambda^{-1}$  means the opposite. All other scenarios result in  $m_n = 1$ , so we begin by showing that there can be no equilibrium in this case.

**CASE 1:**  $m_n = 1 \forall n$ .

A necessary condition for equilibrium when  $m_n = 1$  for both states is a setting where each agent can over-afford his reference point, because every  $\Delta x_{ij}$  must be positive.

Given the tables above, it is also necessary that  $q_1 < w_1, y_{11} < y_{21}$ , and  $y_{12} > y_{22}$  to allow for each agent to over-afford the reference in each state.

This information allows us to re-evaluate the constant,  $k_n$ . If every agent in every state is experiencing a gain relative to his reference point, then the first order conditions are nearly identical across agents. Specifically,

$$m_n = 1 \forall n \rightarrow \Delta x_{11} = \left[ \frac{\theta_1}{\theta_2} \right]^{\frac{1}{a-1}} \cdot \Delta x_{12}; \quad \Delta x_{21} = \left[ \frac{\theta_1}{\theta_2} \right]^{\frac{1}{a-1}} \cdot \Delta x_{22},$$

where  $k_1 = k_2 = (\theta_2 / \theta_1)^{\frac{1}{a-1}}$ , which we will denote simply as  $k$  for the remainder of this case. Taking the ratio of these conditions allows for further simplification of our explicit solutions for the demand deviates:

$$\frac{\Delta x_{11}}{\Delta x_{21}} = \frac{w_2 \cdot (1 + \beta)}{w_2 \cdot (1 + \beta) + \beta - 1} = \frac{\Delta x_{12}}{\Delta x_{22}} < 1,$$

where the inequality implies that the agents both exceed their reference points by a larger margin in the high-state. Using similar techniques as obtained our explicit solutions in terms of  $k$ , we obtain:

$$\begin{aligned} \frac{x_{11} - r_1}{x_{21} - r_1} &= \frac{x_{12} - r_2}{x_{22} - r_2} \rightarrow \\ (x_{11} - r_1)(2\beta y - x_{21} + r_1 - 2w_1 y - 2w_2 \beta y) &= (2y - x_{11} + r_1 - 2w_1 y - 2w_2 \beta y)(x_{21} - r_1) \rightarrow \\ (x_{11} - r_1) \cdot 2y \cdot (\beta - w_1 - w_2) &= (x_{21} - r_1) \cdot 2y \cdot (1 - w_1 - \beta w_2) \rightarrow \\ (x_{11} - r_1)(\beta - 1) &= (x_{21} - r_1)(1 - \beta)w_2 \end{aligned}$$

Now, this finding coupled with our previously established necessary conditions implies that  $0 < \Delta x_{11} = \Delta x_{21} \cdot (-w_2) < 0$ , a contradiction because these demand deviates cannot be both positive and also have opposite sign.

**CASE 2:**  $m_n \neq 1$ .

Even when the agents are not both experiencing a gain or loss, the above relies on the simple relationship between all four of the demand deviates:

$$\frac{\Delta x_{11}}{\Delta x_{21}} = \left( \frac{m_1}{m_2} \right)^{\frac{1}{a-1}} \cdot \frac{\Delta x_{12}}{\Delta x_{22}}$$

where if  $m_1 = m_2$  then the derivations above hold, implying  $\Delta x_{11} = -w_2 \cdot \Delta x_{21}$ , which

holds up to a constant coefficient,  $m = m_1 / m_2$ , as  $\Delta x_{11} = -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{21}$  in all cases.

This necessarily implies the same for agent 2:  $\Delta x_{12} = -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{22}$ . Thus, the

relationship between  $m_1$  and  $m_2$  governs whether agents have a negative

relationship between their demand deviates across states. The following table

outlines the resulting value of  $m$  over the possible pairs,  $(m_1, m_2)$ .

Values of $m$		$m_1$		
		1	$-\lambda$	$-\lambda^{-1}$
$m_2$	1	1	$-\lambda$	$-\lambda^{-1}$
	$-\lambda$	$-\lambda^{-1}$	1	$\lambda^{-2}$
	$-\lambda^{-1}$	$-\lambda$	$\lambda^2$	1



Note that all of the diagonal cells where  $m = 1$  are also ruled out by the above analysis, because they imply that  $\Delta x_{11}$  and  $\Delta x_{21}$  have the same sign, when they cannot. It is also impossible that either  $m_1$  or  $m_2$  is equal to 1, but not both, because this implies  $m < 0$ , so one agent will have only gains or only losses, while the other will have a mixture ( $\Delta x_{1j}, \Delta x_{2j}$  having opposite signs) – a contradiction because  $m < 0$  implies neither agent has a mixture of gains and losses, since in general  $\Delta x_{1j} = -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{2j}$ . Alternatively, this could have been demonstrated by recognizing that  $(1/a - 1)$  induces a non-integer exponent on  $m$ , so if  $m < 0$  then no real roots exist and hence no combination of real-valued  $\Delta x_{nj}$ 's offer a solution. Thus, only two sub-cases remain (highlighted in the table), where  $1 \neq m > 0$ , so one agent gains and the other loses in each state, and a gain for an agent in one state implies he experiences a loss in the other.

These two sub-cases still require the basic relationship,  $\Delta x_{1j} = -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{2j}$ , but we will show that this violates the principle of financial assets in zero net supply. That is, we will show that these demand relationships imply that markets do not clear by substituting in the market clearing constraint for agent 2's demand deviates, given agent 1 facing  $\Delta x_{11} = -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{21}$ :

$$\begin{aligned}
\Delta x_{12} &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot \Delta x_{22} \\
\rightarrow x_{12} - r_2 &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot (x_{22} - r_2) \\
\rightarrow 2y - x_{11} - r_2 &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot (2y\beta - x_{21} - r_2) \\
\rightarrow 2y - x_{11} - r_2 + r_1 - r_1 &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot (2y\beta - x_{21} - r_2 + r_1 - r_1) \\
\rightarrow 2y - \Delta x_{11} - (r_2 + r_1) &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot (2y\beta - \Delta x_{21} - (r_2 + r_1)) \\
\rightarrow 2y - (w_1 \cdot 2y + w_2 \cdot 2y\beta) &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot (2y\beta - (w_1 \cdot 2y + w_2 \cdot 2y\beta)) \\
\rightarrow w_2 \cdot (2y + 2y\beta) &= -w_2 \cdot m^{\frac{1}{a-1}} \cdot 2y \cdot w_1 \cdot (\beta - 1) < 0
\end{aligned}$$

This is, of course, a contradiction because  $0 < w_2 \cdot (2y + 2y\beta)$ . Thus, we have shown there is no equilibrium in the 2x2 model outside of the degenerate one, by identifying all candidate equilibria and then by showing that their component necessary conditions could not simultaneously hold true. It remains to check that the degenerate (non-trading) equilibrium proposed is in-fact an equilibrium.

We defined previously the degenerate equilibrium as the case of perceived actuarially fair prices ( $\mathbf{q} = \mathbf{w}$ ), rendering all agents' reference points exactly equal in cost to their endowments, and enabling them to hold a cash portfolio where they receive exactly the reference point with certainty, experiencing neither gain nor loss in any state. Two questions still remain: do markets clear, and do agents have an incentive to deviate? Consider the market clearing condition,  $\mathbf{q} \cdot \mathbf{x} = \mathbf{q} \cdot \mathbf{y}$ . If agents exactly afford their reference points, then demands must satisfy:

$$\begin{aligned}
q_1 x_{1j} + q_2 x_{2j} &= q_1 \cdot (w_1 y_{1j} + w_2 y_{2j}) + q_2 \cdot (w_1 y_{1j} + w_2 y_{2j}) \\
&= w_1^2 y_{1j} + w_1 w_2 y_{2j} + w_1 w_2 y_{1j} + w_2^2 y_{2j} \\
&= w_1 (w_1 + w_2) y_{1j} + w_2 (w_1 + w_2) y_{2j} \\
&= q_1 y_{1j} + q_2 y_{2j}
\end{aligned}$$

for all agents  $j$ , essentially by construction. Now consider the incentive to deviate. Given the consumption plan of exactly achieving the reference point, agents achieve a stable consumption of zero in all states by removing the gain-loss utility. Without loss, consider a deviation for agent 1 to a new bundle  $\{\hat{x}_{n1}\}$ , where state 1 is demanded at a higher level (it becomes a gain) and state 2 becomes a loss. The difference in the value function is given by:

$$V(\hat{x}) - V(r) = V(\hat{x}) = w_1 \cdot (\hat{x}_{11} - r_1)^a - \lambda \cdot w_2 \cdot (r_1 - \hat{x}_{21})^a.$$

Given that the budget constraint still binds, we know that any increase in state 1 demand will need to be paid for by a decrease in state 2 demand, since we are in the world of perceived actuarially fair prices, so  $\Delta \hat{x}_{11} = -\frac{w_2}{w_1} \Delta \hat{x}_{21}$ . Thus, for the new bundle, we find that there is an incentive to deviate by increasing state 1 consumption, according to the following condition:

$$\begin{aligned} V(\hat{x}) &= w_1 \cdot (\Delta \hat{x}_{11})^a - \lambda \cdot w_2 \cdot \left( \frac{w_2}{w_1} \Delta \hat{x}_{11} \right)^a \\ &= (\Delta \hat{x}_{11})^a \cdot w_1^{-a} \cdot (w_1^{1+a} - \lambda \cdot w_2^{1+a}) > 0 \quad \text{iff} \quad w_1 > w_2 \cdot \lambda^{\frac{1}{1+a}}. \end{aligned}$$

The problem, of course, is that the choice of agent was arbitrary, meaning that both agents have an incentive to deviate towards consuming more in the *same* state, depending on the relative probabilities of states 1 and 2. Substituting in the budget constraint means they can “afford” to do so, so the only way to restore equilibrium would be to adjust prices. Unfortunately, we have already shown that no other equilibrium prices exist. Thus, we have reduced to a truly degenerate equilibrium condition: *for any starting endowments,  $\mathbf{y}$ , the reference dependent financial market*

*is in equilibrium if and only if the prices are perceived actuarially fair ( $\mathbf{q} = \mathbf{w}$ ) and the*

*perceived probabilities are given by  $w_1 = w_2 \cdot \lambda^{\frac{1}{1+a}}$ .*

If we recall the transformation from  $p_n$  to  $w_n$  above, the restriction on suitable probabilities transforms to  $p_1^\gamma = p_2^\gamma \cdot \lambda^{\frac{1}{1+a}}$ , which is equivalent to:

$$\frac{p_1}{p_2} = \lambda^{\frac{1}{\gamma(1+a)}}.$$

To say the least, it seems highly unlikely that exogenous states of nature have any predictable connection to the various parameters of the value function.

### **Discussion.**

Thus far, we have established some apparent violations of expected utility in observed options prices, and motivated a model of reference dependent preferences by examining price discrepancies relative to actuarially fair. Unfortunately, once the problem of properly assessing a stochastic reference point was addressed, we showed that no financial exchange equilibrium can exist, outside of one where all prices are perceived actuarially fair (according to the distorted probability weighting of the agents) and where the physical probabilities coincide with the parameters of the value function according to a specific functional relationship. This all begs the question, what part of the approach is wrong?

Extending the theoretical results from prior research, the initial intuition was that the purchaser of a put option or the seller of a call option will accept prices systematically different than the actuarially fair prices, according to his risk

preferences. For example, a risk-neutral party will never buy above actuarially fair prices, while a risk-averse party will buy puts or sell calls at an expected loss, and a risk-loving investor will do the opposite. Thus, there is a strict relationship enforced between the concavity/convexity of the utility function and the direction of the price distortion away from actuarially fair. Extending these concepts to loss-averse investors with reference dependent preferences, we are still convinced that agents can be considered risk-averse or risk-loving in various regions about their reference points, and accordingly, that their willingness to pay (or to accept) relative to perceived actuarially fair prices is dictated similarly.

We chose the results of prospect theory for our model as a positive approach – the predictions of willingness to pay or accept certain price distortions from fair are in line with the observed distortions in actual financial markets. Thus it seems the problem in our analysis may arise from relying on equilibrium concepts. Specifically, while exchange-traded options markets are relatively complete and free of arbitrage, we have no *ex ante* reason to believe that these markets exist in a stable equilibrium – one in which agents complete trades and are satisfied. In particular, the problem described above where agents reach a portfolio and then have an incentive to deviate may likely occur in real time during market trading. Thus, an obvious conclusion here is that ex-equilibrium analysis (where markets do not necessarily clear all at once) may be most appropriate for evaluating the role of reference dependent preferences in determining prices.

Rabin and Thaler (2001) give an oft-cited plumage argument: nice plumage on a dead parrot does no-one any good. Traditional assumptions of concavity are

nice because they get us (among other things) equilibrium. However, if concavity is the wrong approach, then a wrong equilibrium doesn't help us to understand the underlying truth. If correctly modeled behavior doesn't generate equilibrium, then maybe equilibrium isn't worth much in terms of predicting financial market outcomes.

Future research should consider two approaches to this problem, since revealing preferences through market data will ultimately require a positive theory which matches observed market outcomes, as well as better data to remove some of the muddying factors in current financial markets data. Thus, we recommend an approach of agent-based modeling to develop a comparative dynamics analysis to agents responding to prices and other signals. Simultaneously, an experimental approach is needed to capture financial markets data without transaction costs, information asymmetries, market power, and the like.

On the experimental side, we outline the following approach to generating clean financial data for the purposes of revealing risk preferences. An artificial futures and options market would be created around a random process as an underlying asset, where the probability distribution of the random process is announced in advance. All players compete with virtual trading dollars, subject to constraints on maximum losses and forfeiture of prizes obtainable with virtual dollars at the game's conclusion. To parallel the goal of such markets operating in place of crop insurance, players are initially randomly assigned ownership of 'assets' whose value follows the random process, and all other assets are options, which must have a buyer and seller so they are in zero net supply. The game is thus

designed to be replicable as a real-money financial market for future research, subject to overcoming the challenge of providing free ownership of assets to some participants.

## **References**

Yet to be included...